

# FOCAL RADIUS, RIGIDITY, AND LOWER CURVATURE BOUNDS

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**ABSTRACT.** We show that the focal radius of any submanifold  $N$  of positive dimension in a manifold  $M$  with sectional curvature greater than or equal to 1 does not exceed  $\frac{\pi}{2}$ . In the case of equality, we show that  $N$  is totally geodesic in  $M$  and the universal cover of  $M$  is isometric to a sphere or a projective space with their standard metrics, provided  $N$  is closed.

Our results also hold for  $k^{\text{th}}$ -intermediate Ricci curvature, provided the submanifold has dimension  $\geq k$ . Thus in a manifold with Ricci curvature  $\geq n - 1$ , all hypersurfaces have focal radius  $\leq \frac{\pi}{2}$ , and space forms are the only such manifolds where equality can occur, if the submanifold is closed.

To prove these results, we develop a new comparison lemma for Jacobi fields that exploits Wilking's transverse Jacobi equation.

A Riemannian manifold  $M$  has  $k^{\text{th}}$ -intermediate Ricci curvature  $\geq l$  if for any orthonormal  $(k + 1)$ -frame  $\{v, w_1, w_2, \dots, w_k\}$ , the sectional curvature sum,  $\sum_{i=1}^k \sec(v, w_i)$ , is  $\geq l$  ([32], [26]). For brevity we write  $\text{Ric}_k M \geq l$ . Motivated by Myers theorem we show that if  $\text{Ric}_k M \geq k$ , then all submanifolds with dimension  $\geq k$  have focal radius  $\leq \frac{\pi}{2}$ .

**Theorem A.** *Let  $M$  be a complete Riemannian  $n$ -manifold with  $\text{Ric}_k \geq k$  and  $N$  be any submanifold of  $M$  with  $\dim(N) \geq k$ .*

1. *Every unit speed geodesic  $\gamma$  that leaves  $N$  orthogonally at time 0 has at least  $\dim(N) - k + 1$  focal points for  $N$  in  $[-\frac{\pi}{2}, \frac{\pi}{2}]$ , counting multiplicities. In particular, the focal radius of  $N$  is  $\leq \frac{\pi}{2}$ .*
2. *If the focal radius of  $N$  is  $\frac{\pi}{2}$ , then  $N$  is totally geodesic.*

Since  $\text{Ric}_1 M \geq l$  means that all sectional curvatures of  $M$  are  $\geq l$  and  $\text{Ric}_{n-1} M \geq l$  means that  $M$  has Ricci curvature  $\geq l$ , the theorem applies to  $N \subset M$  if either the Ricci curvature of  $M$  is  $\geq n - 1$  and  $N$  is a hypersurface, or the sectional curvature of  $M$  is  $\geq 1$  and  $\dim(N) \geq 1$ .

We emphasize that  $N$  need not be closed or even complete, and there is no hypothesis about its second fundamental form. On the other hand, if  $N$  happens to be closed and have focal radius  $\frac{\pi}{2}$ , then we determine  $M$  up to isometry.

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**Theorem B.** *Let  $M$  be a complete Riemannian  $n$ -manifold with  $\text{Ric}_k \geq k$ . If  $M$  contains a closed, embedded, submanifold  $N$  with focal radius  $\frac{\pi}{2}$  and  $\dim(N) \geq k$ , then the universal cover of  $M$  is isometric to the sphere or a projective space with the standard metrics, and  $N$  is totally geodesic in  $M$ .*

It is reasonable to compare the Ricci curvature versions of Theorems A and B with the Bonnet-Myers Theorem and Cheng's Maximal Diameter Theorem (cf also Theorem 3 in [5] and Theorem 1 in [9]). While an analogy can be made between the sectional curvature version of Theorem B and the Diameter Rigidity Theorem ([13],[31]), the following example shows that Theorem B applies to more nonsimply connected manifolds.

**Example C.** *Let  $\mathbb{S}^3$  be the unit sphere in  $\mathbb{C} \oplus \mathbb{C}$ , and embed  $\mathbb{S}^1$  as the unit circle in the first copy of  $\mathbb{C}$ . Let  $Q$  be the quaternion group of order 8 in  $SO(4)$ . Then the focal radius of  $N = Q(\mathbb{S}^1)/Q$  in  $M = \mathbb{S}^3/Q$  is  $\frac{\pi}{2}$ , providing an example of Theorem B in which  $N$  is its own focal set. On the other hand,  $M$  has diameter strictly smaller than  $\frac{\pi}{2}$ .*

More generally, let  $\pi : \mathbb{S}^n \rightarrow \mathbb{S}^n/G$  be the quotient map of a properly discontinuous action by  $G$  on  $\mathbb{S}^n$ , and let  $N$  be any closed geodesic in  $\mathbb{S}^n/G$ . Then  $\pi^{-1}(N)$  is the disjoint union of closed geodesics in  $\mathbb{S}^n$ , and hence both  $\pi^{-1}(N)$  and  $N$  have focal radius  $\frac{\pi}{2}$ .

Theorem B implies that the standard unit metric is the only one on any topological sphere with sectional curvature  $\geq 1$  that has a closed submanifold with focal radius  $\frac{\pi}{2}$ . In contrast, the conclusion of the Diameter Rigidity Theorem is softer, since there are many metrics on  $\mathbb{S}^n$  with curvature  $\geq 1$  and diameter  $\geq \frac{\pi}{2}$ , and there is even the possibility of such a metric on an exotic sphere.

It is also reasonable to compare the sectional curvature version of Theorem B to the ‘‘rank rigidity’’ results of Schmidt and Shankar–Spatzier–Wilking in [23] and [25]. Shankar, Spatzier, and Wilking obtained the conclusion of Theorem B for manifolds with curvature *less* than or equal to 1 and minimal conjugate radius  $\pi$ . Schmidt proves that if  $M$  has sectional curvature  $\geq 1$  and conjugate radius  $\geq \frac{\pi}{2}$ , then its universal cover is homeomorphic to  $S^n$  or isometric to a projective space. The conjugate radius hypotheses of these theorems apply to every geodesic in  $M$ . In contrast, the focal radius hypothesis of Theorem B only concerns the geodesics that meet a single submanifold orthogonally.

Next we give examples showing that the hypotheses about the dimension of the submanifolds in the Theorems above cannot be removed. For the sectional curvature versions of the theorems, a point in small perturbation of  $\mathbb{S}^n$  shows that the conclusions can be false if  $N$  does not have positive dimension. For the Ricci curvature versions of the theorems, we have the following examples.

**Example D.** *Let  $S_k^n$  be the  $n$ -sphere with constant curvature  $k$ . The product metric on  $S_{\frac{n+1}{n-1}}^n \times S_{n+1}^2$  satisfies*

$$\begin{aligned} \text{Ric} \left( S_{\frac{n+1}{n-1}}^n \times S_{n+1}^2 \right) &= n+1 = \text{Ric}(\mathbb{S}^{n+2}), \text{ and} \\ \text{FocalRadius}(\{pt\} \times S_{n+1}^2) &= \pi \sqrt{\frac{n-1}{n+1}} \rightarrow \pi \text{ as } n \rightarrow \infty. \end{aligned} \tag{0.0.1}$$

Thus the focal radius of  $N$  in the Ricci curvature version of Theorem A can converge to  $\pi$  if the hypothesis that  $N$  is a hypersurface is removed and the dimension of  $M$  is allowed to go to  $\infty$ , while the dimension of  $N$  is fixed.

On the other hand, if we take  $n = 2$  or  $3$ , then 0.0.1 becomes

$$\begin{aligned} Ric(S_3^2 \times S_3^2) &\equiv 3 \equiv Ric(\mathbb{S}^4), \\ \text{FocalRadius}(\{pt\} \times S_3^2) &= \pi \sqrt{\frac{1}{3}} > \frac{\pi}{2}, \text{ and} \\ Ric(S_2^3 \times S_4^2) &= 4 = Ric(\mathbb{S}^5), \\ \text{FocalRadius}(S_2^3 \times \{pt\}) &= \frac{\pi}{2}. \end{aligned}$$

So the hypothesis that  $N$  is a hypersurface in Ricci curvature versions of Theorems A and B cannot be replaced with the hypothesis that  $N$  is a codimension 2 submanifold.

For our intermediate Ricci curvature results we have

**Example E.** For  $k > \frac{4}{3}p$  and  $p \geq 2$ ,  $M = S_{\frac{k}{k-p}}^{k-1} \times S_k^p$  satisfies

$$\begin{aligned} Ric_k(M) &\geq k \text{ and} \\ \text{FocalRadius}(\{pt\} \times S_k^p) &= \pi \sqrt{\frac{k-p}{k}} > \frac{\pi}{2}, \text{ if } k > \frac{4}{3}p. \end{aligned}$$

Thus  $\{pt\} \times S_k^p \subset S_{\frac{k}{k-p}}^{k-1} \times S_k^p$  is a closed submanifold of a  $(k+p-1)$ -manifold with  $Ric_k(M) \geq k$  and focal radius  $> \frac{\pi}{2}$ , and the focal radius of  $N$  in Theorem A can exceed  $\frac{\pi}{2}$  if the hypothesis that  $\dim(N) \geq k$  is replaced with  $\dim(N) \geq p$  where  $\frac{3}{4}k > p$ .

By sending  $k \rightarrow \infty$  while keeping  $p$  fixed, we see that

$$\text{FocalRadius}(\{pt\} \times S_k^p) = \pi \sqrt{\frac{k-p}{k}} \rightarrow \pi.$$

So in Theorem A, the focal radius of  $N$  can converge to  $\pi$ , if there is no hypothesis about the dimension of  $N$ , and the dimension of  $M$  is allowed to go to  $\infty$ .

To prove Theorems A and B, we exploit Wilking's transverse Jacobi equation ([30]) to get a new comparison lemma for Jacobi fields. To state it, we let  $\gamma : (-\infty, \infty) \rightarrow M$  be a unit speed geodesic in a complete Riemannian  $n$ -manifold  $M$ . We call an  $(n-1)$ -dimensional subspace  $\Lambda$  of normal Jacobi fields along  $\gamma$ , *Lagrangian*, if the restriction of the Riccati operator to  $\Lambda$  is self adjoint, that is, if

$$\langle J_1(t), J_2'(t) \rangle = \langle J_1'(t), J_2(t) \rangle$$

for all  $t$  and for all  $J_1, J_2 \in \Lambda$  (see 1.0.8 below for the formal definition of the Riccati operator on  $\Lambda$ ).

In Section 1, we review Wilking's transverse Jacobi equation, justify the name Lagrangian, and prove a comparison lemma for intermediate Ricci curvature. In the special case when the sectional curvature is bounded from below our comparison result becomes the following.

**Lemma F.** (*Sectional Curvature Comparison*) For  $\kappa = -1, 0$ , or  $1$ , let  $\gamma : (-\infty, \infty) \rightarrow M$  be a unit speed geodesic in a complete Riemannian  $n$ -manifold  $M$  with  $\sec(\dot{\gamma}, \cdot) \geq \kappa$ . Let  $J_0$  be a nonzero, normal Jacobi field along  $\gamma$ , and let  $\Lambda$  be a Lagrangian subspace of normal Jacobi fields along  $\gamma$  with Riccati operator  $S$  such that  $J_0 \in \Lambda$ .

For  $t_0 < t_{\max}$ , suppose that  $\Lambda$  has no singularities on  $(t_0, t_{\max})$ , and that  $\tilde{\lambda}_\kappa : [t_0, t_{\max}) \rightarrow \mathbb{R}$  is a solution of

$$\tilde{\lambda}'_\kappa + \tilde{\lambda}_\kappa^2 + \kappa = 0 \quad (0.0.2)$$

with

$$\langle S(J_0), J_0 \rangle|_{t_0} \leq \tilde{\lambda}_\kappa(t_0) |J_0(t_0)|^2. \quad (0.0.3)$$

Then for each  $t_1 \in [t_0, t_{\max})$  there is a  $J_1 \in \Lambda \setminus \{0\}$  so that

$$\langle S(J_1), J_1 \rangle|_{t_1} \leq \tilde{\lambda}_\kappa(t_1) |J_1(t_1)|^2. \quad (0.0.4)$$

In particular, if  $\kappa = 1$ ,  $\alpha \in [0, \pi)$ ,  $\tilde{\lambda}_1(t) = \cot(t + \alpha)$ , and  $t_0 \in [0, \pi - \alpha)$ , then  $\Lambda$  has a singularity by time  $\pi - \alpha$ , that is, there is a  $J \in \Lambda \setminus \{0\}$  with  $J(t_2) = 0$  for some  $t_2 \in (t_0, \pi - \alpha]$ .

**Remark.** Lemma F also holds when  $t_0$  is a singular value of  $\tilde{\lambda}_\kappa$  with  $\lim_{t \rightarrow t_0^+} \tilde{\lambda}_\kappa(t) = \infty$ .

The reader is probably familiar with the Riccati comparison theorem of Eschenburg-Heintze in [8]. It requires the initial condition (0.0.3) to hold for *all*  $J_0 \in \Lambda$ , while Lemma F only demands that the initial condition holds for a *single* Jacobi field. This comes at the expense that the derived future inequality (0.0.4) is only guaranteed to hold for a single Jacobi field, which moreover, is not likely to be the original field. In Example 2.7 (below), we show that  $J_1$  can in fact be different from  $J_0$ . A similar example can be found on page 463 of [17]. This phenomenon is tied to the nonvanishing of Wilking's generalized  $A$ -tensor (see (1.4.2)).

The difference between Lemma F and the theorem of [8] is starker if one considers the contrapositives: Lemma F implies that if Inequality 0.0.4 fails for all  $J_1 \in \Lambda$ , then Inequality 0.0.3 fails for *all*  $J \in \Lambda$ . In contrast, the theorem of [8] only gives that Inequality 0.0.3 fails for *some*  $J \in \Lambda$ .

The main tool to prove Theorem A is Lemma 1.2, which is a generalization of Lemma F to intermediate Ricci curvature. So that we can prove Theorem B, Lemma 1.2 also includes an analysis of the rigid situation. For the convenience of the reader we detail in Corollary 1.3 the rigidity case of Lemma F, that is we explain what happens when Inequality 0.0.4 is an equality.

The proof of Theorem B begins by establishing Proposition 3.3, which draws a strong analogy between  $N$  and one of the dual sets in the proof of the Diameter Rigidity Theorem. Example C shows that we can only push this analogy so far. The dual sets of [13] are disjoint while Example C shows that  $N$  can be its own focal set. In fact, one of the challenges of the proof of Theorem B is showing that phenomena like Example C do not occur in the simply connected case. In spite of the differences, our overall strategy is similar to that of [13], and our proof employs ideas from there. To keep the exposition tight, we will often refrain from giving further specific references to [13] and have made our exposition reasonably self-contained.

In the course of proving Theorem B we will also establish the following corollary (see in Theorem 4.8, below.)

**Corollary G.** *If the submanifold  $N$  of Theorem B is a hypersurface, then the universal cover of  $M$  is isometric to the unit sphere.*

After the introduction, we establish notations and conventions. The remainder of the paper is divided into two parts and 7 sections. The sections are subordinate to the parts. Each part and many of the sections begin with a detailed summary of the contents, so the outline immediately below is only meant to indicate where each result is proven.

Part 1 contains Sections 1 and 2. In Section 1, we review Riccati comparison and Wilking's transverse Jacobi equation, and we state Lemma 1.2, which is the main tool of the paper. We prove Theorem A and Lemma 1.2 in Section 2. In subsection 2.1, we provide an example that shows that  $J_0$  and  $J_1$  can indeed be different in Lemma F.

In Part 2, we prove Theorem B in Sections 3—7. In the special case of Theorem B, when the sectional curvature is  $\geq 1$ , the argument can be completed a little faster by an appeal to the Diameter Rigidity Theorem. We do this in Section 6, and we complete the proof of Theorem B in Section 7.

**Remark H.** *The reader may have noticed that the hypotheses  $\text{Ric}_k \geq k \cdot \kappa$  of Theorems A and B are global, whereas in Lemma F, we only assumed that  $\sec(\dot{\gamma}, \cdot) \geq \kappa$ .*

*For the conclusion of Theorems A to hold, we in fact, only need  $\text{Ric}_k(\dot{\gamma}, \cdot) \geq k \cdot \kappa$  for all unit speed geodesics  $\gamma$  that leave  $N$  orthogonally at time 0. That is,*

$$\sum_{i=1}^k \sec(\dot{\gamma}, E_i) \geq k \cdot \kappa$$

*for any orthonormal set  $\{\dot{\gamma}, E_1, \dots, E_k\}$ .*

*On the other hand, our proof of Theorem B uses the global hypothesis  $\text{Ric}_k \geq k \cdot \kappa$  and also the fact that Lemma F and Corollary 1.3 are valid with only the radial curvature lower bound.*

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## NOTATIONS AND CONVENTIONS

Unless otherwise specified, all curves are parameterized at unit speed. Given  $v \in TM$ , we denote the unique geodesic with  $\gamma'_v(0) = v$  by  $\gamma_v$ .

Let  $N$  be a submanifold of the Riemannian manifold  $M$ . Let  $\nu(N)$  be the normal bundle of  $N \subset M$ . For every unit  $v \in \nu(N)$ , there is a first time  $t_1 \in (0, \infty]$  at which  $\gamma_v(t_1)$  is focal for  $N$  along  $\gamma_v$ . We set

$$\text{reg}_N \equiv \{tv \in \nu(N) \mid |v| = 1 \text{ and } t \in [0, t_1)\}.$$

We let  $g^*$  be the metric on the domain  $\text{reg}_N$  obtained from pulling back  $(M, g)$  via the normal exponential map. We use the term *tangent focal point* for a critical point of  $\exp_N^\perp : \nu(N) \rightarrow M$  and the term *focal point* for a critical value of  $\exp_N^\perp$ .

$\pi : \nu(N) \longrightarrow N$  will denote the projection of the normal bundle;  $N_0$  will be the 0-section of  $\nu(N)$ , and  $\nu^1(N)$  will be the unit normal bundle of  $N$ . The fibers of  $\nu(N)$  and  $\nu^1(N)$  over  $x \in N$  will be called  $\nu_x(N)$  and  $\nu_x^1(N)$ .

We let  $\Lambda$  be any Lagrangian family of normal Jacobi fields along a geodesic  $\gamma$ , and for any subspace  $L \subset \Lambda$  we write

$$L(t) \equiv \{J(t) \mid J \in L\} \oplus \{J'(t) \mid J \in L \text{ and } J(t) = 0\}. \quad (0.0.5)$$

When  $\gamma$  is a geodesic that leaves  $N$  orthogonally at time 0, we will write  $\Lambda_N$  for the Lagrangian family of normal Jacobi fields along  $\gamma$  corresponding to variations by geodesics that leave  $N$  orthogonally at time 0. According to Lemma 4.1 on page 227 of [6],  $\Lambda_N$  consists of the following *normal* Jacobi fields  $J$  along  $\gamma$ :

$$\Lambda_N \equiv \{J \mid J(0) = 0, J'(0) \in \nu_{\gamma(0)}(N)\} \oplus \{J \mid J(0) \in T_{\gamma(0)}N \text{ and } J'(0) = S_{\gamma'(0)}J(0)\}, \quad (0.0.6)$$

where  $S_{\gamma'(0)}$  is the shape operator of  $N$  determined by  $\gamma'(0)$ , that is,

$$\begin{aligned} S_{\gamma'(0)} &: T_{\gamma_v(0)}N \longrightarrow T_{\gamma_v(0)}N \text{ is} \\ S_{\gamma'(0)} &: w \longmapsto (\nabla_w \gamma'(0))^{TN}. \end{aligned}$$

Unless otherwise specified, we assume that  $\Lambda$  is nonsingular on  $(t_0, t_{\max})$ . That is, the evaluation map  $\mathcal{E}_t : \Lambda \longrightarrow T_{\gamma(t)}M$ ,  $\mathcal{E}_t(J) = J(t)$  is one-to-one for all  $t \in (t_0, t_{\max})$ .

We write  $\mathbb{S}^n$  for the unit sphere in  $\mathbb{R}^{n+1}$ , and for  $\kappa = -1, 0$ , or  $1$ , we let  $\mathcal{S}_\kappa^2$  be the simply connected 2-dimensional space form of constant curvature  $\kappa$ .

We use the acronym CROSS for Compact Rank One Symmetric Space. For convenience, we normalize the nonspherical CROSSes so that their curvatures are in  $[1, 4]$ , and we normalize the spherical CROSSes to have constant curvature 4.

We write  $\sec$  for sectional curvature and  $\kappa$  for our lower curvature bound. After rescaling, we may always assume that  $\kappa$  is either  $-1, 0$ , or  $1$ .

Given  $r > 0$  and  $A \subset M$  we set

$$\begin{aligned} B(A, r) &\equiv \{x \in M \mid \text{dist}(x, A) < r\}, \\ D(A, r) &\equiv \{x \in M \mid \text{dist}(x, A) \leq r\}, \text{ and} \\ S(A, r) &\equiv \{x \in M \mid \text{dist}(x, A) = r\}. \end{aligned}$$

Finally, we write  $D_v(f)$  the derivative of  $f$  in the direction  $v$ .

## Part 1. Bounding the Focal Radius

Part 1 is divided into Sections 1 and 2. In Section 1, we state Lemma 1.2, which is a generalization of Lemma F and is the main tool of the paper. We also review Lagrangian families, Wilking's transverse Jacobi equation, and Riccati comparison.

In Section 2, we prove Lemma 1.2 and Theorem A, and in Subsection 2.1 we give an example that shows that  $J_1$  need not equal  $J_0$  in Lemma F.

### 1. JACOBI FIELD COMPARISON

In this section, we review Lagrangian families, Wilking's transverse Jacobi equation, and Riccati comparison.

**1.1. Lagrangian Families.** Let  $\gamma$  be a unit speed geodesic in a complete Riemannian  $n$ -manifold  $M$ , and let  $\mathcal{J}$  be the vector space of normal Jacobi fields along  $\gamma$ . Using symmetries of the curvature tensor, we see that for  $J_1, J_2 \in \mathcal{J}$ ,

$$\omega(J_1, J_2) = \langle J'_1, J_2 \rangle - \langle J_1, J'_2 \rangle,$$

is constant along  $\gamma$  and hence defines a symplectic form on  $\mathcal{J}$ .

Thus an  $(n-1)$ -dimensional subspace  $\Lambda$  of  $\mathcal{J}$  on which  $\omega$  vanishes is called Lagrangian. Of course this is equivalent to saying that the restriction of the Riccati operator to  $\Lambda$  is self-adjoint. Examples of Lagrangian families include the Jacobi fields that are 0 at time 0 and those that correspond to variations by geodesics that leave a submanifold orthogonally at time 0.

The set of times  $t$  so that

$$\{J(t) \mid J \in \Lambda\} = \text{span}\{\dot{\gamma}(t)\}^\perp \quad (1.0.7)$$

is open and dense (cf Lemma 1.7 of [14]). For these  $t$  we get a well-defined Riccati operator

$$\begin{aligned} S_t &: \text{span}\{\dot{\gamma}(t)\}^\perp \longrightarrow \text{span}\{\dot{\gamma}(t)\}^\perp \\ S_t &: v \longmapsto J'_v(t), \end{aligned} \quad (1.0.8)$$

where  $J_v$  is the unique  $J_v \in \Lambda$  so that  $J_v(t) = v$ . The Jacobi equation then decomposes into the two first order equations

$$S(J) = J', \quad S' + S^2 + R = 0,$$

where  $S'$  is the covariant derivative of  $S$  along  $\gamma$  and  $R$  is the curvature along  $\gamma$ , that is  $R(\cdot) = R(\cdot, \dot{\gamma})\dot{\gamma}$  (see Equation 1.7.1 in [14]).

**Remark 1.1.** For fixed  $t \in \mathbb{R}$ , let

$$\begin{aligned} \mathcal{E}_t &: \Lambda \longrightarrow T_{\gamma(t)}M \\ \mathcal{E}_t &: J \longmapsto J(t) \end{aligned}$$

be the evaluation map. Given any  $W \subset \Lambda$  for which  $\mathcal{E}_t|_W$  is one-to-one, Equation 1.0.8 gives a well defined Riccati operator  $W(t) \longrightarrow \gamma'(t)^\perp$ .

**1.2. Statements of Comparison Lemmas.** For  $\kappa = -1, 0$ , or  $1$ , let  $\mathcal{S}_\kappa^2$  be the simply connected 2-dimensional space form of constant curvature  $\kappa$ . Let  $\tilde{\Lambda}$  be a Lagrangian subspace of normal Jacobi fields along a geodesic  $\tilde{\gamma}$  in  $\mathcal{S}_\kappa^2$ , let

$$\tilde{\lambda}_\kappa(t) \text{ be the eigenvalue of the Riccati operator of } \tilde{\Lambda} \text{ at time } t, \quad (1.1.1)$$

and let  $\tilde{f}$  be a function so that

$$\tilde{\Lambda} = \text{span}\{\tilde{f}\tilde{E}\}, \quad (1.1.2)$$

where  $\tilde{E}$  is a parallel field along  $\tilde{\gamma}$ .

The Lagrangian subspaces in  $\mathcal{S}_\kappa^2$  are the spans of single normal Jacobi fields, so the possibilities for  $\tilde{f}$  are

$$\tilde{f}(t) = \begin{cases} (c_1 \sin t + c_2 \cos t) & \text{if } \kappa = 1, \\ (c_1 t + c_2) & \text{if } \kappa = 0, \\ (c_1 \sinh t + c_2 \cosh t) & \text{if } \kappa = -1, \end{cases} \quad (1.1.3)$$

where  $c_1, c_2 \in \mathbb{R}$ , and the possibilities for  $\tilde{\lambda}_\kappa$  are the logarithmic derivatives of the functions in 1.1.3. These are given explicitly on page 302 of [7]. They are also solutions of the ODE

$$\tilde{\lambda}'_\kappa + \tilde{\lambda}_\kappa^2 + \kappa = 0$$

of Equation 0.0.2.

For a subspace  $W \subset \Lambda$ , write

$$W(t) = \{J(t) \mid J \in W\} \oplus \{J'(t) \mid J \in W \text{ and } J(t) = 0\},$$

and

$$P_{W,t} : \Lambda(t) \longrightarrow W(t)$$

for orthogonal projection. For simplicity of notation we will write

$$\text{Trace}(S(t)|_W) \text{ for } \text{Trace}(P_{W,t} \circ S(t)|_W).$$

By considering 1-dimensional subspaces, we see that Lemma F is a special case of the following result.

**Lemma 1.2.** (*Intermediate Ricci Comparison*) For  $\kappa = -1, 0$ , or  $1$ , let  $\gamma : (-\infty, \infty) \longrightarrow M$  be a unit speed geodesic in a complete Riemannian  $n$ -manifold  $M$  with  $\text{Ric}_k \geq k \cdot \kappa$ . Let  $\Lambda$  be a Lagrangian subspace of normal Jacobi fields along  $\gamma$  with Riccati operator  $S$ . For  $t_0 < t_{\max}$ , suppose that  $\Lambda$  has no singularities on  $(t_0, t_{\max})$  and that for some  $k$ -dimensional subspace  $W_{t_0} \subset \Lambda$  and some Lagrangian subspace  $\tilde{\Lambda}$  along  $\tilde{\gamma}$  in  $\mathcal{S}_\kappa^2$ ,

$$\text{Trace}(S(t_0)|_{W_{t_0}}) \leq k \cdot \tilde{\lambda}_\kappa(t_0), \quad (1.2.1)$$

where  $\tilde{\lambda}_\kappa$  is as in (1.1.1). Then the following statements hold.

1. For all  $t \in [t_0, t_{\max})$  there is a  $k$ -dimensional subspace  $W_t \subset \Lambda$  so that

$$\text{Trace}(S(t)|_{W_t}) \leq k \cdot \tilde{\lambda}_\kappa(t). \quad (1.2.2)$$

2. For  $t \in [t_0, t_{\max})$ , set

$$\{W_{t_0}\}^\perp(t) \equiv \{L \in \Lambda \mid L(t) \perp W_{t_0}(t)\}.$$

If for some  $t_1 \in (t_0, t_{\max})$ ,

$$\{W_{t_0}\}^\perp(t_1) = \{W_{t_0}\}^\perp(t_0)$$

and

$$\text{Trace}(S(t_1)|_{W_{t_0}}) = k \cdot \tilde{\lambda}_\kappa(t_1),$$

then  $W_{t_0}$  consists of Jacobi fields whose restrictions to  $[t_0, t_1]$  have the form

$$J = \tilde{f} \cdot E,$$



where  $E$  is a unit parallel field and  $\tilde{f}$  is the function from 1.1.2 that satisfies  $\tilde{f}(t_0) = |J(t_0)|$ . Moreover, for  $t \in (t_0, t_1]$ , the subspaces

$$W_t^\perp \equiv \{J \in \Lambda \mid J(t) \perp W_{t_0}(t)\}$$

are independent of  $t$ . In particular,  $\Lambda$  splits orthogonally as

$$\Lambda \equiv W_{t_0} \oplus W_{t_0}^\perp.$$

3. The conclusion of Part 2 holds, with  $t_1 = t_{\max}$ , if  $\lim_{t \rightarrow t_{\max}^-} \tilde{\lambda}_\kappa(t) = -\infty$  and  $J(t) \neq 0$  for all  $J \in \Lambda \setminus \{0\}$  and all  $t \in (t_0, t_{\max})$ .

4. If  $\kappa = 0$ ,  $t_1 = t_{\max} = \infty$ ,  $\tilde{\lambda}_\kappa \equiv 0$ , and  $J(t) \neq 0$  for all  $J \in \Lambda \setminus \{0\}$  and all  $t \in (t_0, \infty)$ , then the conclusion of Part 2 holds.

In the special case when the sectional curvature is bounded from below of Part 1 of Lemma 1.2 becomes Lemma F. Similarly, Parts 2 and 3 of Lemma 1.2 have the following corollary for sectional curvature.

**Corollary 1.3.** (Sectional Curvature Rigidity) For  $\kappa = -1, 0$ , or  $1$ , let  $\gamma : (-\infty, \infty) \rightarrow M$  be a unit speed geodesic in a complete Riemannian  $n$ -manifold  $M$  with  $\sec(\dot{\gamma}, \cdot) \geq \kappa$ . Let  $J_0$  be a nonzero, normal Jacobi field along  $\gamma$ , and let  $\Lambda$  be a Lagrangian subspace of normal Jacobi fields along  $\gamma$  with Riccati operator  $S$  such that  $J_0 \in \Lambda$ .

For  $t_0 < t_{\max}$ , suppose that  $\Lambda$  has no singularities on  $(t_0, t_{\max})$ , and for some Lagrangian subspace  $\tilde{\Lambda}$  along  $\tilde{\gamma}$  in  $\mathcal{S}_\kappa^2$ ,

$$\langle S(J_0), J_0 \rangle|_{t_0} \leq \tilde{\lambda}_\kappa(t_0) |J_0(t_0)|^2.$$

1. For  $t \in [t_0, t_{\max})$ , set

$$\{J_0\}^\perp(t) \equiv \{L \in \Lambda \mid L(t) \perp J_0(t)\}.$$

If for some  $t_1 \in (t_0, t_{\max})$ ,

$$\{J_0\}^\perp(t_1) = \{J_0\}^\perp(t_0)$$

and

$$\langle S(J_0), J_0 \rangle|_{t_1} = \tilde{\lambda}_\kappa(t_1) |J_0(t_1)|^2,$$

then for all  $s \in [t_0, t_1]$ ,

$$\langle S(J_0), J_0 \rangle|_s = \tilde{\lambda}_\kappa(s) |J_0(s)|^2, \tag{1.3.1}$$

and the subspaces  $\{J_0\}^\perp(s)$  are independent of  $s \in [t_0, t_1]$ . Moreover, on  $[t_0, t_1]$ ,

$$J_0(s) = \tilde{f}(s) E,$$

where  $E$  is a unit parallel field and  $\tilde{f}$  is the function from 1.1.2 that satisfies  $\tilde{f}(t_0) = |J_0(t_0)|$ .

2. The conclusion of Part 1 holds with,  $t_1 = t_{\max}$ , if  $\lim_{t \rightarrow t_{\max}^-} \tilde{\lambda}_\kappa(t) = -\infty$  and  $J(s) \neq 0$  for all  $J \in \Lambda \setminus \{0\}$  and all  $s \in (t_0, t_{\max})$ .

3. If  $\kappa = 0$ ,  $t_1 = t_{\max} = \infty$ ,  $\tilde{\lambda}_\kappa \equiv 0$ , and  $J(t) \neq 0$  for all  $J \in \Lambda \setminus \{0\}$  and all  $t \in (t_0, \infty)$ , then the conclusion of Part 1 holds.

In the remainder of this section we review background material that we will employ in Section 2 to prove Lemma 1.2.

**1.3. Wilking's Transverse Jacobi Equation.** Let  $\mathcal{V}$  be any subspace of  $\Lambda$ . Set

$$V(t) \equiv \{J(t) \mid J \in \mathcal{V}\} \oplus \{J'(t) \mid J \in \mathcal{V}, J(t) = 0\}. \quad (1.3.2)$$

Then  $V(t)$  is a smooth vector bundle along  $\gamma$  (Lemma 1.7.1 in [14], or [30]). Set

$$H(t) \equiv V(t)^\perp \cap \dot{\gamma}(t)^\perp.$$

**Proposition 1.4.** *Suppose that the kernel of the evaluation map,  $\mathcal{E}_t$ , lies in  $\mathcal{V}$ .*

1. *For  $x \in H(t)$ , there is a  $J \in \Lambda$  so that  $J(t) = x$ .*

2. *We have a well-defined Riccati operator*

$$\hat{S} : H(t) \longrightarrow H(t)$$

*given by*

$$\hat{S}(x) = \left( (J^H)' \right)^H(t), \quad (1.4.1)$$

*where  $J$  is an element of  $\Lambda$  so that  $J(t) = x$ , and  $J^H(t)$  is the  $H(t)$ -component of  $J(t)$ .*

*Proof.* Since  $\Lambda$  is Lagrangian, the splitting

$$\Lambda(t) = \{J(t) \mid J \in \Lambda\} \oplus \{J'(t) \mid J \in \Lambda, J(t) = 0\}$$

is orthogonal. Since the kernel of  $\mathcal{E}_t$  lies in  $\mathcal{V}$ ,  $H(t)$  is contained in the first summand, and Part 1 follows.

For the second part, suppose  $x = J(t) + K(t)$ , where  $K \in \text{Kernel}(\mathcal{E}_t)$ . Since  $\text{Kernel}(\mathcal{E}_t) \subset \mathcal{V}$ ,

$$\left( \left( (J + K)^H \right)' \right)^H \Big|_t = \left( (J^H)' \right)^H \Big|_t,$$

as claimed. □

Wilking also defined maps

$$\begin{aligned} A_t & : V(t) \longrightarrow H(t) \text{ given by,} \\ A_t(v) & = (J')^h(t), \text{ where } J \in \mathcal{V}, J(t) = v. \end{aligned} \quad (1.4.2)$$

A priori,  $A_t$  is only defined at points where  $V(t) = \{J(t) \mid J \in \mathcal{V}\}$ ; however,  $A$  extends smoothly to  $\mathbb{R}$  (cf. [30]). Indeed, let  $A_t^* : H(t) \longrightarrow V(t)$  be the adjoint of  $A_t$ , and let  $X$  be a field in  $H$  so that  $(X')^H \equiv 0$ . According to Equation 1.7.6 on page 38 of [14],

$$X' = -A^*(X).$$

Since the left-hand side is smooth,  $A^*$  is smooth, and it follows that  $A$  is smooth.

Like the Gray-O'Neill  $A$ -tensor, the Wilking  $A$ -tensor vanishes identically along a geodesic  $\gamma$  if and only if the distributions  $V(t)$  and  $H(t)$  are parallel along  $\gamma$ . In this case, it follows that

$$\{J \in \Lambda \mid J(t) \in H(t)\}$$

is independent of  $t$ , and the parallel, orthogonal splitting  $V(t) \oplus H(t)$  is given by Jacobi fields.

**Theorem 1.5.** (*Wilking, [30]*)  $\hat{S}$  is self-adjoint, and

$$\hat{S}' + \hat{S}^2 + \{R(\cdot, \dot{\gamma})\dot{\gamma}\}^h + 3AA^* = 0. \quad (1.5.1)$$

**Remark 1.6.** Equation 1.5.1 is known as the Transverse Jacobi Equation. It is a vast generalization of the Horizontal Curvature Equation of [11] and [20]. For details see [15] or [18].

**Remark 1.7.** Proposition 1.4 only gives us that  $\hat{S}$  is defined almost everywhere. However, since  $\{R(\cdot, \dot{\gamma})\dot{\gamma}\}^h + 3AA^*$  is smooth everywhere, it follows that  $\hat{S}' + \hat{S}^2$  has a smooth extension to all of  $\mathbb{R}$ . (See [30] for an interpretation of  $\hat{S}' + \hat{S}^2$  as a second order differential operator  $H(t) \rightarrow H(t)$ .)

**1.4. Riccati Comparison.** In this subsection, we review the Riccati comparison results of Eschenburg ([7]) and Eschenburg–Heintze ([8]).

**Theorem 1.8.** (*Eschenburg–Heintze, [8], cf Proposition 2.3 in [7]*) Let  $r : \mathbb{R} \rightarrow \mathbb{R}$  be a  $C^\infty$ -function with  $r \geq \kappa$ . For  $\kappa = 0$  or  $-1$  let  $t_0 \in \mathbb{R}$  and for  $\kappa = 1$ , let  $t_0 \in [0, \pi]$ . Let  $s$  be a smooth solution of the initial value problem

$$s' + s^2 + r = 0, \quad s(t_0) \leq \tilde{\lambda}_\kappa(t_0)$$

on the interval  $[t_0, t_{\max})$ , where  $\tilde{\lambda}_\kappa$  is as in 1.1.1. Then

1.

$$s(t) \leq \tilde{\lambda}_\kappa(t) \quad (1.8.1)$$

on  $[t_0, t_{\max})$ .

2. If  $s(t_1) = \tilde{\lambda}_\kappa(t_1)$  for some  $t_1 \in (t_0, t_{\max})$ , then for all  $t \in [t_0, t_1]$

$$s(t) = \tilde{\lambda}_\kappa(t) \quad \text{and} \quad r|_{[t_0, t_1]} \equiv \kappa. \quad (1.8.2)$$

When  $s$  is the trace of the Riccati operator of a Lagrangian family in  $\text{Ric} \geq \kappa(n-1)$ , the rigidity of Part 2 in Theorem 1.8 also yields rigidity of  $S$  and  $R(\cdot, \dot{\gamma})\dot{\gamma}$ . This idea goes back at least as far as the Splitting Theorem ([3]) and Cheng's Maximal Diameter Theorem, ([4]). It also appears in Croke and Kleiner's paper on rigidity of warped products ([5]), in Theorem 1.7.1 of [14], and in Theorem H of [15]. Since our applications will be to the transverse Jacobi equation, we formulate it in terms of abstract Riccati equations.

**Lemma 1.9.** For  $t \in [t_0, t_{\max})$ , let  $\hat{S}(t), \hat{R}(t) : V \rightarrow V$  be symmetric endomorphisms of a  $k$ -dimensional vector space  $V$  so that

$$\hat{S}' + \hat{S}^2 + \hat{R} = 0.$$

Let  $\tilde{\lambda}_\kappa$  be as in 1.1.1 and have no singularities on  $(t_0, t_{\max})$ , and suppose that

$$\text{Trace}(\hat{S})(t_0) \leq k \cdot \tilde{\lambda}_\kappa(t_0), \quad \text{and} \quad (1.9.1)$$

$$\text{Trace}(\hat{R})(t) \geq k \cdot \kappa$$

for all  $t \in [t_0, t_{\max})$ . Then

$$\hat{S} \equiv \tilde{\lambda}_\kappa \cdot \text{id} \text{ and } \hat{R} = \kappa \cdot \text{id} \quad (1.9.2)$$

if any of the following hold.

1. If

$$\lim_{t \rightarrow t_{\max}^-} \tilde{\lambda}_\kappa(t) = -\infty,$$

then 1.9.2 holds on  $[t_0, t_{\max})$ .

2. If  $t_{\max} = \infty$ ,  $\kappa = 0 = \tilde{\lambda}_0(t_0)$ , then 1.9.2 holds on  $[t_0, \infty)$ , that is  $\hat{R} \equiv \hat{S} \equiv 0$ .

3. If  $\text{Trace} \hat{S}(t_1) = k \tilde{\lambda}_\kappa(t_1)$  for some  $t_1 \in (t_0, t_{\max}]$ , then 1.9.2 holds on  $[t_0, t_1]$ .

In particular, if the hypotheses of any of 1, 2, or 3 hold, then the solutions of the Jacobi equation  $J'' + \hat{R}(J) = 0$  on  $(t_0, t_{\max})$ ,  $(t_0, \infty)$ , or  $(t_0, t_1)$ , respectively, have the form

$$J = \tilde{f}E, \quad (1.9.3)$$

where  $E$  is a constant vector in  $V$  and  $\tilde{f}$  is the function from 1.1.2 that satisfies  $\tilde{f}(t_0) = |J(t_0)|$ .

*Proof.* Set

$$\begin{aligned} s &\equiv \frac{1}{k} \text{Trace}(\hat{S}), \\ \hat{S}_0 &\equiv \hat{S} - \frac{\text{Trace}(\hat{S})}{k} \cdot \text{id}, \text{ and} \\ r &\equiv \frac{1}{k} \left( \text{Trace}(\hat{R}) + |\hat{S}_0|^2 \right). \end{aligned} \quad (1.9.4)$$

Taking the trace of

$$\hat{S}' + \hat{S}^2 + \hat{R} = 0$$

yields

$$s' + s^2 + r = 0.$$

By 1.9.1 and 1.8.1,

$$s(t) \leq \tilde{\lambda}_\kappa(t) \quad (1.9.5)$$

for all  $t \in (t_0, t_{\max})$ . For Part 1, we have  $\lim_{t \rightarrow t_{\max}^-} \tilde{\lambda}_\kappa(t) = -\infty$ , so

$$\tilde{\lambda}_\kappa(t) = \begin{cases} \cot(\pi + t - t_{\max}) & \text{if } \kappa = 1, \\ \frac{1}{t - t_{\max}} & \text{if } \kappa = 0, \\ \coth(t - t_{\max}) & \text{if } \kappa = -1 \end{cases}$$

(see, e.g., page 302 of [7]). Since  $\tilde{\lambda}_\kappa$  has no singularities on  $(t_0, t_{\max})$ , it follows that  $\tilde{\lambda}_\kappa(t)$  is strictly decreasing on  $(t_0, t_{\max})$ . So if Inequality 1.9.5 is strict for some  $t_1 \in (t_0, t_{\max})$ , then there is an  $\alpha \in (0, t_{\max} - t_1)$  so that

$$s(t_1) \leq \tilde{\lambda}_\kappa(t_1 + \alpha).$$

Thus by 1.8.1,

$$s(t) \leq \tilde{\lambda}_\kappa(t + \alpha)$$

for all  $t \in (t_1, t_{\max})$ . In particular, for some  $\tilde{t}_{\max} \in (t_0, t_{\max} - \alpha]$ ,  $\lim_{t \rightarrow \tilde{t}_{\max}^-} s(t) = -\infty$ . Since this contradicts our hypothesis that  $\hat{S}$  is defined on  $(t_0, t_{\max})$ , Inequality 1.9.5 must be an equality for all  $t \in (t_0, t_{\max})$ . In particular, to complete the proof of Part 1, it suffices to prove Part 3.

For Part 2, since  $\kappa = 0 = \tilde{\lambda}_0(t_0)$ , 1.8.1 gives

$$s(t) \leq 0 \quad (1.9.6)$$

for all  $t \in [t_0, \infty)$ . If Inequality 1.9.6 is strict for some  $t_1 \in [t_0, \infty)$ , then

$$s(t_1) < \tilde{\lambda}_0(t_1),$$

where

$$\tilde{\lambda}_0(t) = \frac{1}{t - c}$$

for some  $c > t_1$ . By 1.8.1,

$$s(t) \leq \tilde{\lambda}_0(t)$$

for all  $t \in [t_1, c)$ . In particular, for some  $\tilde{c} \in (t_1, c]$ ,  $\lim_{t \rightarrow \tilde{c}^-} s(t) = -\infty$ . Since this contradicts our hypothesis that  $\hat{S}$  is defined on  $[t_0, \infty)$ , Inequality 1.9.6 must be an equality for all  $t \in (t_0, t_{\max})$ . In particular, to complete the proof of Part 2, it suffices to prove Part 3.

For part 3, first observe that Equation 1.8.2 gives us  $s(t) \equiv \tilde{\lambda}_\kappa(t)$  and  $r \equiv \kappa$ .

Consequently,

$$\begin{aligned} \kappa &= r \\ &= \frac{\text{Trace}(\hat{R}) + |\hat{S}_0|^2}{k} \\ &\geq \frac{\kappa k + |\hat{S}_0|^2}{k} \\ &= \kappa + \frac{|\hat{S}_0|^2}{k}. \end{aligned}$$

Thus  $|\hat{S}_0| \equiv 0$  and

$$\begin{aligned} \hat{S} &= \frac{\text{trace}(\hat{S})}{k} \cdot \text{id} \\ &= s \cdot \text{id} \\ &= \tilde{\lambda}_\kappa(t) \cdot \text{id}. \end{aligned}$$

Substituting  $\hat{S} = \tilde{\lambda}_\kappa(t) \cdot \text{id}$  into the Riccati equation,  $\hat{S}^2 + \hat{S}' + \hat{R} = 0$ , gives

$$\begin{aligned} \left( \tilde{\lambda}_\kappa(t)^2 + \tilde{\lambda}_\kappa'(t) \right) \cdot \text{id} + \hat{R} &= 0, \\ -\kappa \cdot \text{id} + \hat{R} &= 0, \text{ and} \\ \hat{R} &= \kappa \cdot \text{id}. \end{aligned}$$

So the Jacobi fields have the form 1.9.3.  $\square$

**Remark 1.10.** *In the event that  $\lim_{t \rightarrow t_0^+} \tilde{\lambda}_\kappa(t) = \infty$ , both Theorem 1.8 and Lemma 1.9 hold with the hypothesis  $s(t_0) = \frac{1}{k} \text{Trace}(\hat{S}) \leq \tilde{\lambda}_\kappa(t_0)$  replaced by*

$$\liminf_{t \rightarrow t_0^+} \left( \tilde{\lambda}_\kappa(t) - s(t) \right) \geq 0. \quad (1.10.1)$$

*If  $s$  is the trace of the Riccati operator of the Lagrangian family  $\{J \mid J(t_0) = 0\}$  along a geodesic in a Riemannian manifold, then Inequality 1.10.1 is satisfied with*

$$\tilde{\lambda}_\kappa(t) = \begin{cases} \cot t & \text{if } \kappa = 1 \\ \frac{1}{t} & \text{if } \kappa = 0 \\ \coth t & \text{if } \kappa = -1 \end{cases}$$

*(see Theorem 27 on page 175 of [22]). So, for example, in this case, Theorem 1.8 implies the classical Rauch Comparison Theorem for 2-manifolds.*

## 2. FOCAL RADIUS AND POSITIVE CURVATURE

In this section, we combine Riccati comparison with the Transverse Jacobi Equation to prove Lemma 1.2 and Theorem A, and in Subsection 2.1, we give an example to show that the fields  $J_0$  and  $J_1$  in Lemma F can indeed be different.

**Lemma 2.1.** *(Eigen Transfer Lemma) Let  $\gamma : [0, l] \rightarrow M$  and  $\Lambda$  be as in Lemma 1.2. Let  $\mathcal{V}$  be an  $(n - 1 - k)$ -dimensional subspace of  $\Lambda$  so that for all  $t \in [0, l]$ , the kernel of the evaluation map*

$$\begin{aligned} \mathcal{E}_t &: \Lambda \longrightarrow T_{\gamma(t)}M \\ \mathcal{E}_t &: J \longmapsto J(t) \end{aligned}$$

*lies in  $\mathcal{V}$ . For any subspace  $L$  of  $\Lambda$  define  $L(t)$  as in (0.0.5).*

*1. For each fixed  $\bar{t} \in [0, l]$ , there is an  $k$ -dimensional subspace  $W$  of  $\Lambda$  so that  $W(\bar{t})$  is the orthogonal complement of  $\mathcal{V}(\bar{t})$ . If  $\mathcal{E}_{\bar{t}}$  is one-to-one, then  $W$  is unique.*

*2. Let  $\hat{S}$  be as in 1.4.1. Then for any  $W$  as in Part 1,*

$$\text{Trace}(\hat{S})|_{\bar{t}} \leq k \cdot \lambda$$

*if and only*

$$\text{Trace}(S|_W)|_{\bar{t}} \leq k \cdot \lambda,$$

*where  $\text{Trace}(S|_W)|_{\bar{t}} = \text{Trace}(P_{W, \bar{t}} \circ S|_{W(\bar{t})})$ , and  $P_{W, \bar{t}} : \Lambda(\bar{t}) \rightarrow W(\bar{t})$  is orthogonal projection.*

**Remark 2.2.** *For any  $W$  as in Part 1,  $S|_W$  is well defined via Remark 1.1.*

*Proof.* If  $\mathcal{E}_{\bar{t}}$  happens to be one-to-one, then it is an isomorphism onto  $\mathcal{V}(\bar{t})^\perp$ , so the existence of a unique  $W$  as in Part 1 is clear. Otherwise, let  $K \in \Lambda$  be in the kernel of  $\mathcal{E}_{\bar{t}}$ . Then for  $J \in \Lambda$

$$\langle J, K' \rangle|_{\bar{t}} = \langle J', K \rangle|_{\bar{t}} = 0.$$

It follows that  $\mathcal{E}_{\bar{t}} : \Lambda \longrightarrow T_{\gamma(\bar{t})}M$  maps onto the orthogonal complement of

$$\{K'(\bar{t}) \mid K \in \ker(\mathcal{E}_{\bar{t}})\}.$$

Since  $\ker(\mathcal{E}_{\bar{t}}) \subset \mathcal{V}$ , it follows that  $\text{im}(\mathcal{E}_{\bar{t}})$  contains the orthogonal complement of

$$\mathcal{V}(\bar{t}) = \{J(\bar{t}) \mid J \in \mathcal{V}\} \oplus \{K'(\bar{t}) \mid K \in \mathcal{V} \text{ and } K(\bar{t}) = 0\},$$

and Part 1 follows.

To prove Part 2, for  $J \in W$ , we write

$$J^\perp = J - J^\mathcal{V},$$

where  $J^\mathcal{V}$  is the component of  $J$  that lies in  $\mathcal{V}(t)$ . Then for all  $t$ ,

$$0 = \frac{d}{dt} \langle J^\mathcal{V}, J^\perp \rangle = \langle (J^\mathcal{V})', J^\perp \rangle + \langle J^\mathcal{V}, J^{\perp'} \rangle.$$

Since  $J \in W$ ,  $J^\mathcal{V}(\bar{t}) = 0$ , and  $\langle J^\mathcal{V}, J^{\perp'} \rangle|_{\bar{t}} = 0$ . So the previous display evaluated at  $\bar{t}$  becomes

$$\langle (J^\mathcal{V})', J^\perp \rangle|_{\bar{t}} = 0.$$

For  $J \in W$ , it follows that

$$\begin{aligned} \langle \hat{S}(J^\perp), J^\perp \rangle|_{\bar{t}} &= \langle (J' - (J^\mathcal{V})'), J^\perp \rangle|_{\bar{t}} \\ &= \langle J', J^\perp \rangle|_{\bar{t}} \\ &= \langle S(J), J^\perp \rangle|_{\bar{t}} \\ &= \langle S(J), J \rangle|_{\bar{t}}. \end{aligned} \tag{2.2.1}$$

So

$$\text{Trace}(\hat{S})|_{\bar{t}} \leq k \cdot \lambda$$

if and only if

$$\text{Trace}(S|_W)|_{\bar{t}} \leq k \cdot \lambda.$$

□

*Proof of Lemma 1.2.* We combine Theorem 1.8 and Lemma 1.9 with the Transverse Jacobi Equation, and the Eigen Transfer Lemma 2.1.

Start by setting

$$\mathcal{V} \equiv \{X \in \Lambda \mid X(t_0) \perp J(t_0) \text{ for all } J \in W_{t_0}\}.$$

Let  $\hat{S} : H(t) \longrightarrow H(t)$  be as in Equation 1.4.1. It follows from the Eigen Transfer Lemma 2.1 that

$$\text{Trace}(\hat{S})|_{t_0} \leq k \cdot \tilde{\lambda}_\kappa(t_0).$$

The Transverse Jacobi Equation says,

$$\hat{S}' + \hat{S}^2 + \{R(\cdot, \dot{\gamma}(t)) \dot{\gamma}(t)\}^h + 3AA^* = 0. \tag{2.2.2}$$

Since  $\text{Ric}_k \geq k$ ,  $AA^*$  is nonnegative, and  $W_{t_0}$  is  $k$ -dimensional, when we take the trace of Equation 2.2.2, divide by  $k$ , and make the substitutions of 1.9.4, we get an equation that satisfies the hypotheses of Theorem 1.8. Thus for all  $t \in [t_0, t_{\max})$ ,

$$\frac{1}{k} \text{Trace} \hat{S}(t) \leq \tilde{\lambda}_\kappa(t). \quad (2.2.3)$$

For  $t \in [t_0, t_{\max})$ , set

$$W_t \equiv \{J \in \Lambda \mid J(t) \perp \mathcal{V}(t)\}.$$

By combining 2.2.3 with the Eigen Transfer Lemma 2.1 and the lack of singularities of  $\Lambda|_{(t_0, t_{\max})}$ , we have

$$\text{Trace}(S|_{W_t})|_t \leq k \cdot \lambda_\kappa,$$

as claimed.

To prove Part 2, suppose that

$$\{W_{t_0}\}^\perp(t_1) = \{W_{t_0}\}^\perp(t_0) \text{ and } \text{Trace}(S(t_1)|_{W_{t_0}}) = k \cdot \tilde{\lambda}_\kappa(t_1)$$

for some  $t_1 \in (t_0, t_{\max})$ .

It follows from Equation 2.2.1 that

$$\text{Trace} \hat{S}(t_1) = k \cdot \tilde{\lambda}_\kappa(t_1).$$

Writing

$$\hat{R} \text{ for } \{R(\cdot, \dot{\gamma}(t)) \dot{\gamma}(t)\}^h + 3AA^*,$$

we see from Theorem 1.8 that

$$\text{Trace} \hat{S}(t) \equiv k \cdot \tilde{\lambda}_\kappa(t) \text{ and } \text{Trace}(\hat{R}) \equiv k \cdot \kappa$$

for all  $t \in [t_0, t_1]$ .

Our hypothesis that  $\text{Ric}_k \geq k \cdot \kappa$  implies that  $\text{Trace} \{R(\cdot, \dot{\gamma}(t)) \dot{\gamma}(t)\}^h \geq k \cdot \kappa$ . Combining this with  $\text{Trace}(\hat{R}) \equiv k \cdot \kappa$  and the fact that  $AA^*$  is nonnegative, we see that  $A \equiv 0$ . So

$$W_t \equiv \{J \in \Lambda \mid J(t) \perp \mathcal{V}(t)\}$$

is independent of  $t$ , and  $W_t = W_{t_0}$  for all  $t$ . It follows that  $\Lambda$  splits orthogonally as

$$\Lambda \equiv W_{t_0} \oplus \mathcal{V}.$$

By Part 3 of Lemma 1.9,  $\hat{S} \equiv \tilde{\lambda}_\kappa \cdot \text{id}$  and  $\hat{R} = \kappa \cdot \text{id}$ . So it follows that  $W_{t_0}$  consists of Jacobi fields whose restrictions to  $[t_0, t_1]$  have the form

$$J = \tilde{f}E,$$

where  $E$  is a parallel field and  $\tilde{f}$  is the function from 1.1.2 that satisfies  $\tilde{f}(t_0) = |J(t_0)|$ .

To prove Part 3, we suppose that  $\Lambda$  has no singularities on  $[t_0, t_{\max})$ , so by Proposition 1.4,  $\hat{S}$  is defined on  $[t_0, t_{\max})$ . As above, the Eigen Transfer Lemma 2.1 gives us that

$$\text{Trace}(\hat{S})|_{t_0} \leq k \cdot \tilde{\lambda}_\kappa(t_0).$$



So by Part 1 of Lemma 1.9,  $\hat{S} \equiv \tilde{\lambda}_\kappa \cdot \text{id}$  and  $\hat{R} = \kappa \cdot \text{id}$  on  $[t_0, t_{\max})$ . As before, our hypothesis that  $\text{Ric}_k \geq k \cdot \kappa$  implies that  $\text{Trace} \{R(\cdot, \dot{\gamma}(t)) \dot{\gamma}(t)\}^h \geq k \cdot \kappa$ . Combining this with  $\hat{R} \equiv k \cdot \kappa$  and the fact that  $AA^*$  is nonnegative, we see that  $A \equiv 0$ . So

$$W_t \equiv \{J \in \Lambda \mid J(t) \perp \mathcal{V}(t)\}$$

is independent of  $t$ , and  $W_t = W_{t_0}$  for all  $t$ . It follows that  $\Lambda$  splits orthogonally as

$$\Lambda \equiv W_{t_0} \oplus \mathcal{V}.$$

Since  $\hat{S} \equiv \tilde{\lambda}_\kappa \cdot \text{id}$  and  $\hat{R} = \kappa \cdot \text{id}$ , as in Part 2, we have that  $W_{t_0}$  consists of Jacobi fields whose restrictions to  $[t_0, t_1]$  have the form

$$J = \tilde{f}E,$$

where  $E$  is a parallel field and  $\tilde{f}$  is the function from 1.1.2 that satisfies  $\tilde{f}(t_0) = |J(t_0)|$ .

To prove Part 4, we suppose that  $\Lambda$  has no singularities on  $[t_0, \infty)$ . It follows from Proposition 1.4 that  $\hat{S}$  is defined on  $[t_0, \infty)$ . As above, the Eigen Transfer Lemma 2.1 gives us that

$$\text{Trace} \left( \hat{S} \right) \Big|_{t_0} \leq 0.$$

So by Part 2 of Lemma 1.9,  $\hat{S} \equiv \tilde{\lambda}_\kappa \cdot \text{id} \equiv 0$  and  $\hat{R} = \kappa \cdot \text{id} \equiv 0$  on  $[t_0, \infty)$ . As before, our hypothesis that  $\text{Ric}_k \geq 0$  implies that  $\text{Trace} \{R(\cdot, \dot{\gamma}(t)) \dot{\gamma}(t)\}^h \geq 0$ . Combining this with  $\hat{R} \equiv 0$  and the fact that  $AA^*$  is nonnegative, we see that  $A \equiv 0$ . So

$$W_t \equiv \{J \in \Lambda \mid J(t) \perp \mathcal{V}(t)\}$$

is independent of  $t$ , and  $W_t = W_{t_0}$  for all  $t$ . It follows that  $\Lambda$  splits orthogonally as

$$\Lambda \equiv W_{t_0} \oplus \mathcal{V}.$$

Since  $\hat{S} \equiv 0$  and  $\hat{R} = 0$ , as in Part 2, we have that  $W_{t_0}$  consists of Jacobi fields whose restrictions to  $[t_0, t_1]$  have the form

$$J = E,$$

where  $E$  is a parallel field. □

The proof of Lemma 1.2 has the added bonus of choosing  $W_t$  in terms of  $\mathcal{V}(t)$ .

**Corollary 2.3.** *Let  $\mathcal{V}$  be as in the proof of Lemma 1.2. In Part 1 of Lemma 1.2, we can choose the subspace  $W_t \subset \Lambda$  so that*

$$W_t(t) \perp \mathcal{V}(t).$$

**Remark 2.4.** *Notice that our proof of Lemma 1.2 only requires that the radial intermediate Ricci curvatures along  $\gamma$  are bounded from below, that is,*

$$\sum_{i=1}^k \sec(\dot{\gamma}, E_i) \geq k \cdot \kappa$$

for any orthonormal set  $\{\dot{\gamma}, E_1, \dots, E_k\}$ .

**Remark 2.5.** *The proof above shows that Lemma 1.2 is valid even when  $\Lambda$  has singularities on  $(t_0, t_{\max})$ , as long as they lie in  $\mathcal{V}$ . Indeed, Theorem 1.8 holds on intervals where the function  $s$  is smooth. In our context, this happens as long as  $\hat{S}$  is well-defined. According to Proposition 1.4,  $\hat{S}$  is well-defined at all times  $t$  where the kernel of the evaluation map lies in  $\mathcal{V}$ .*

**Remark 2.6.** *If  $\lim_{t \rightarrow t_0^+} \tilde{\lambda}_\kappa(t) = \infty$ , then, using Remark 1.10, Lemma 1.2 holds with the hypothesis  $\text{Trace}(S|_{W_{t_0}}(t)) \leq k\tilde{\lambda}_\kappa(t)$  replaced with*

$$\liminf_{t \rightarrow t_0^+} \left( \text{Trace}(S|_{W_{t_0}}(t)) - k\tilde{\lambda}_\kappa(t) \right) \geq 0, \quad (2.6.1)$$

and

$$\tilde{\lambda}_\kappa(t) = \begin{cases} \cot t & \text{if } \kappa = 1 \\ \frac{1}{t} & \text{if } \kappa = 0 \\ \coth t & \text{if } \kappa = -1. \end{cases}$$

*If  $N$  is a smooth submanifold of  $M$ , then Inequality 2.6.1 holds for*

$$W_0 = \{J | J(0) = 0, J'(0) \in \nu_{\gamma(0)}(N)\} \subset \Lambda_N$$

*(see Part 3 of Lemma 2.7 in [24] and also Remark 3 in [8]).*

*Proof of Theorem A (cf Theorem 3.5 in [10]).* Let  $v \in \nu(N)$  be any unit vector. It suffices to show that for

$$\begin{aligned} \mathcal{K} &\equiv \left\{ J \in \Lambda_N \mid J(t_i) = 0 \text{ for some nonzero } t_i \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \right\}, \\ \dim \mathcal{K} &\geq \dim(N) - k + 1. \end{aligned}$$

The definition of  $\mathcal{K}$  implies that for all  $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ , the kernel of the evaluation map is contained in  $\mathcal{K}$ .

Suppose, by way of contradiction, that  $\dim \mathcal{K} \leq \dim(N) - k$ , and set

$$\mathcal{K}(t) \equiv \{J(t) \mid J \in \mathcal{K}\} \oplus \{J'(t) \mid J \in \mathcal{K} \text{ and } J(t) = 0\}.$$

Since  $\dim \mathcal{K} \leq \dim(N) - k$ , and

$$\dim \{J | J(0) \in T_{\gamma_v(0)}N \text{ and } J'(0) = S_v J(0)\} = \dim(N),$$

there is a  $k$ -dimensional subspace

$$W \subset \{J | J(0) \in T_{\gamma_v(0)}N \text{ and } J'(0) = S_v J(0)\}$$

with

$$W(0) \perp \mathcal{K}(0).$$

Replacing  $\gamma_v$  with  $\gamma_{-v}$  if necessary we may assume that

$$\text{Trace}(S|_W(0)) \leq 0. \quad (2.6.2)$$

Let  $\mathcal{V} \subset \Lambda_N$  be the subspace so that  $\mathcal{V}(0) \perp W(0)$ , and notice that

$$\mathcal{K} \subset \mathcal{V}. \quad (2.6.3)$$

From 2.6.2 and 2.6.3, we see, using Remark 2.5, that Lemma 1.2 applies to  $\Lambda_N$  and  $W$  on  $[0, \frac{\pi}{2}]$ . So for all  $t \in [0, \frac{\pi}{2}]$ , there is a  $k$ -dimensional space  $W_t \subset \Lambda_N$  so that

$$\text{Trace}(S(t)|_{W_t}) \leq k \cot\left(t + \frac{\pi}{2}\right). \quad (2.6.4)$$

By Corollary 2.3,  $W_t(t) \perp \mathcal{V}(t)$ . It follows from this and Inequality 2.6.4 that there is a  $Z \in \Lambda \setminus \mathcal{V}$  with

$$Z(t) = 0 \text{ for some } t \in \left(0, \frac{\pi}{2}\right]. \quad (2.6.5)$$

Since  $Z \notin \mathcal{V}$ , it follows that  $Z \notin \mathcal{K}$ , and 2.6.5 contradicts the definition of  $\mathcal{K}$ .

To prove Part 2, assume that the focal radius of  $N$  is  $\frac{\pi}{2}$ . Apply Part 3 of Lemma 1.2 with  $\kappa = 1$ ,  $t_0 = 0$ ,  $t_{\max} = \frac{\pi}{2}$ , and  $\tilde{\lambda}_1 = \cot\left(t + \frac{\pi}{2}\right)$ , to conclude that

$$\{J|J(0) \in T_{\gamma_v(0)}N \text{ and } J'(0) = S_v J(0)\}$$

is spanned by Jacobi fields of the form  $\sin\left(t + \frac{\pi}{2}\right)E$  where  $E$  is a parallel field. In particular,  $S_v \equiv 0$ , and since this holds for all unit vectors  $v$  orthogonal to  $N$ ,  $N$  is totally geodesic.  $\square$

**2.1. Why  $J_1$  need not be  $J_0$ .** This subsection neither depends on nor is used in the rest of the paper. In it we give an example that shows that the field  $J_1$  in Lemma F can indeed be different from the field  $J_0$ . A similar example can be found on page 463 of [17].

**Example 2.7.** Let  $E_1$  and  $E_2$  be parallel orthonormal fields along a geodesic  $\gamma$  in  $\mathbb{R}^3$  with  $E_1, E_2 \perp \gamma$ . Let  $\Lambda$  be the Lagrangian family

$$\Lambda = \text{span}\{tE_1, (t+1)E_2\}.$$

Let

$$J_0 = tE_1 + (t+1)E_2.$$

Then

$$\langle J'_0(0), J_0(0) \rangle = \tilde{\lambda}_0(0) \langle J_0(0), J_0(0) \rangle = 1,$$

where  $\tilde{\lambda}_0 = \frac{1}{t+1}$  comes from the model Jacobi field on  $\mathbb{R}^2$  given by  $\tilde{J} = (t+1)\tilde{E}$  with  $\tilde{E}$  a parallel field. In particular,  $J_0$  satisfies Inequality 0.0.3 with  $t_0 = 0$ .

On the other hand,

$$\langle J'_0(t), J_0(t) \rangle = \langle E_1 + E_2, tE_1 + (t+1)E_2 \rangle = 2t + 1,$$

and for  $t > 0$ ,

$$\tilde{\lambda}_0(t) \langle J_0(t), J_0(t) \rangle = \frac{1}{t+1} (t^2 + (t+1)^2) = \frac{t^2}{t+1} + t + 1 < 2t + 1 = \langle J'_0(t), J_0(t) \rangle.$$

To verify the validity of Lemma F for this example, take  $J_1(t) = (t+1)E_2$  and note that Inequality 0.0.4 is an equality for all  $t > 0$ .

## Part 2. Focal Rigidity

Let  $M$  be a complete Riemannian manifold  $M$  with  $Ric_k \geq k$ , and let  $N$  be a closed submanifold of  $M$  of dimension at least  $k$  and focal radius  $\frac{\pi}{2}$ . Since each connected component of  $N$  has focal radius  $\frac{\pi}{2}$ , we may assume that  $N$  is connected.

In the second part of the paper, we prove Theorem B by showing that the universal cover of  $M$  is isometric to the unit sphere or to a projective space with the standard metric, with  $N$  totally geodesic in  $M$ .

In Section 3, we exploit the rigidity portion of Lemma 1.2 to prove a rigidity result for the Jacobi fields of  $\Lambda_N$  (see Proposition 3.3). This allows us to prove, in Section 4, that every first focal point of  $N$  is regular in the sense of [16]. With this it follows rather easily that  $F$ , the focal set of  $N$ , is a totally geodesic closed submanifold with focal radius  $\frac{\pi}{2}$ . We thus further the analogy between the pair  $(N, F)$  and the dual sets in the proof of the Diameter Rigidity Theorem. In particular, we establish, as in [13], that  $F$  (resp.  $N$ ) is the base of a Riemannian submersion from the unit normal sphere to any point of  $N$  (resp.  $F$ ). In Section 4, we also show that if  $\dim(F) + \dim(N) = \dim(M) - 1$ , then  $M$  has constant curvature 1, which in particular yields Corollary G.

To show that phenomena like Example C do not occur in the simply connected case, we prove, in Section 5, that our focal set  $F$  is *very regular* in the sense of Hebda ([16]). This allows us to appeal to Theorem 3.1 in [16] and conclude, in Theorem 5.2, that  $M$  is the union of two disk bundles. Using this we prove that if the codimension of  $F$  (resp.  $N$ ) is  $\geq 3$ , then  $N$  (resp.  $F$ ) is simply connected; hence the fibers of the Riemannian submersion to  $N$  (resp.  $F$ ) are connected.

All of the above allows us to complete the proof of Theorem B along the lines of the proof of the Diameter Rigidity Theorem. In the sectional curvature case, the argument can be concluded more rapidly. We prove that the diameter of the universal cover of  $M$  is  $\geq \frac{\pi}{2}$ , and appeal to the Diameter Rigidity Theorem, after making a further topological argument that rules out exotic spheres and nonunit metrics on  $\mathbb{S}^n$ . We give the details of this in Section 6. In Section 7, we complete the proof of Theorem B for intermediate Ricci curvature.

### 3. THE DISTANCE FROM $N$

With the exception of Proposition 3.3, we assume throughout Sections 3—7 that  $M$  is a complete Riemannian manifold with  $Ric_k \geq k$ , and that  $N$  is a connected, closed submanifold of  $M$  of dimension at least  $k$  and focal radius  $\frac{\pi}{2}$ .

In this section, we apply the rigidity statement in Lemma 1.2 to prove Proposition 3.3, which says, among other things, that the radial sectional curvatures from  $N$  are all  $\geq 1$ .

We start by reviewing the notion of horizontally homothetic submersions.

**Definition 3.1.** ([1], [2]) *A submersion  $\pi : M \rightarrow B$  of Riemannian manifolds is called horizontally homothetic if and only if there is a smooth function  $\lambda : M \rightarrow (0, \infty)$  with vertical gradient so that for all horizontal vectors  $x$  and  $y$*

$$\lambda^2 \langle x, y \rangle_M = \langle D\pi(x), D\pi(y) \rangle_B.$$

We also use the following result from [21].

**Proposition 3.2.** *Let  $\pi : M \longrightarrow B$  be a horizontally homothetic submersion with dilation  $\lambda$  and let  $r$  be a regular value of  $\lambda$  so that  $\lambda^{-1}(r)$  is nonempty. Then*

$$\pi|_{\lambda^{-1}(r)} : (\lambda^{-1}(r), \langle \cdot, \cdot \rangle_M) \longrightarrow \left( B, \frac{1}{r^2} \langle \cdot, \cdot \rangle_B \right)$$

*is a Riemannian submersion.*

For a unit speed geodesic  $\gamma_v$  that leaves  $N$  orthogonally at time 0, we set

$$\begin{aligned} \mathcal{Z}_N &\equiv \{ J | J(0) = 0, J'(0) \perp \text{span} \{ T_{\gamma_v(0)}N, \gamma'_v(0) \} \} \\ \mathcal{T}_N &\equiv \{ J | J(0) \in T_{\gamma_v(0)}N \text{ and } J'(0) = S_v J(0) \}, \text{ and} \\ \Lambda_N &\equiv \mathcal{Z}_N \oplus \mathcal{T}_N, \end{aligned} \tag{3.2.1}$$

where  $S_v$  is the shape operator of  $N$  determined by  $v$ , that is,  $S_v : T_{\gamma_v(0)}N \longrightarrow T_{\gamma_v(0)}N$ , is  $(\nabla_v)^{TN}$ .

Our first consequence of  $\text{FocalRadius}(N) = \frac{\pi}{2}$  holds even if  $N$  is not closed, and it only requires that the radial intermediate Ricci curvatures from  $N$  are  $\geq k \cdot \kappa$ . Theorem A and Remark H imply that such an  $N$  is totally geodesic.

**Proposition 3.3.** *Let  $M$  be a complete Riemannian  $n$ -manifold, and let  $N$  be a submanifold of  $M$  with focal radius  $\frac{\pi}{2}$  and  $\dim(N) \geq k$ . Suppose that along each unit speed geodesic  $\gamma : [0, \frac{\pi}{2}] \longrightarrow M$  that leaves  $N$  orthogonally at time 0 we have  $\text{Ric}_k(\dot{\gamma}, \cdot) \geq k \cdot \kappa$ , that is*

$$\sum_{i=1}^k \sec(\dot{\gamma}, E_i) \geq k \cdot \kappa,$$

*for any orthonormal set  $\{\dot{\gamma}, E_1, \dots, E_k\}$ .*

1. *All  $J \in \mathcal{T}_N$  have the form  $J(t) = \cos t E$  where  $E$  is a parallel field along  $\gamma$ .*
2.  *$\mathcal{Z}_N \oplus \mathcal{T}_N$  is a parallel, orthogonal splitting along  $[0, \frac{\pi}{2}]$ .*
3. *Let  $g^*$  be the metric on  $\text{reg}_N \subset \nu(N)$  obtained from pulling back  $(M, g)$  via the normal exponential map, and let  $\pi : \text{reg}_N \longrightarrow N$  be the projection of the normal bundle. Then with respect to  $g^*$ ,  $\pi$  is a horizontally homothetic submersion with scaling function  $\cos(\text{dist}(N_0, \cdot))$ , where  $N_0$  is the 0-section of the normal bundle,  $\nu(N)$ .*
4. *If  $c : I \longrightarrow N$  is a unit speed geodesic in  $N$ , and  $V$  is a parallel normal unit field along  $c$ , then*

$$\Phi : I \times \left(0, \frac{\pi}{2}\right) \longrightarrow M, \quad \Phi(s, t) = \exp_{c(s)}^\perp(tV(s))$$

*is a totally geodesic immersion whose image has constant curvature 1.*

5. *With respect to  $g^*$ , every plane tangent to  $\text{reg}_N \setminus N_0$  that contains  $X \equiv \text{grad} \{ \text{dist}(N_0, \cdot) \}$  has sectional curvature  $\geq 1$ .*

*Proof.* Parts 1 and 2 follow by combining Remark 2.4 with the version of Part 3 of Lemma 1.2 when  $\kappa = 1$ ,  $t_0 = 0$ ,  $t_{\max} = \frac{\pi}{2}$ , and  $\tilde{\lambda}_1 = \cot(t + \frac{\pi}{2})$ . (Cf also Theorem B in [15].)

For Part 3, we let  $\mathcal{Z}_N^*$  and  $\mathcal{T}_N^*$  be the pullbacks of  $\mathcal{Z}_N$  and  $\mathcal{T}_N$  to  $\text{reg}_N$  via  $\exp_N^\perp$ . Observe that by Part 2,

$$\text{grad}\{\text{dist}(N_0, \cdot)\} \oplus \mathcal{Z}_N^* \oplus \mathcal{T}_N^*$$

is an orthogonal splitting of  $T\text{reg}_N$  with respect to  $g^*$ . Since the vertical space of  $\pi$  is spanned by  $\text{grad}\{\text{dist}(N_0, \cdot)\} \oplus \mathcal{Z}_N^*$ , the horizontal space of  $\pi$  with respect to  $g^*$  is spanned by  $\mathcal{T}_N^*$ . Since the fields of  $\mathcal{T}_N^*$  come from variations of geodesics that leave  $N_0$  orthogonally, they are  $\pi$ -basic. Part 3 follows by combining this with Part 1.

For Part 4, observe that Part 1 gives us that  $\Phi$  is an immersion. Let  $\text{II}$  be the second fundamental form of image  $(\Phi)$ . By construction,  $\frac{\partial \Phi}{\partial t}$  is a geodesic field so  $\text{II}\left(\frac{\partial \Phi}{\partial t}, \frac{\partial \Phi}{\partial t}\right) = 0$ . It follows from Part 1 that  $\text{II}\left(\frac{\partial \Phi}{\partial t}, \frac{\partial \Phi}{\partial s}\right) = 0$ .

To see that  $\text{II}\left(\frac{\partial \Phi}{\partial s}, \frac{\partial \Phi}{\partial s}\right) = 0$ , we note that, since  $\frac{\partial \Phi}{\partial t}$  is normal to  $\exp_N^\perp(S(N_0, r))$ , it suffices to verify that

$$g\left\langle \text{II}\left(\frac{\partial \Phi}{\partial s}, \frac{\partial \Phi}{\partial s}\right), Z \right\rangle = 0 \quad (3.3.1)$$

for all  $Z \in T\exp_N^\perp(S(N_0, r))$  that are normal to the image of  $\Phi$ . Since  $\exp_N^\perp : (\text{reg}_N, g^*) \rightarrow (M, g)$  is a local isometry, to prove 3.3.1, it suffices to do the corresponding calculation in  $(\text{reg}_N, g^*)$ . From Part 3 we have that the restriction of  $\pi : (\text{reg}_N, g^*) \rightarrow \left(N, \frac{1}{\cos(t)^2}g\right)$  to the  $t$ -level set of  $\text{dist}(N_0, \cdot)$  is a Riemannian submersion. Let  $\widetilde{\frac{\partial \Phi}{\partial s}}$  be a lift of  $\frac{\partial \Phi}{\partial s}$  to  $\text{reg}_N$  via  $\exp_N^\perp$ . Then  $\widetilde{\frac{\partial \Phi}{\partial s}}$  is a  $\pi$ -basic horizontal, geodesic field, so  $\text{II}\left(\frac{\partial \Phi}{\partial s}, \frac{\partial \Phi}{\partial s}\right) = (D\exp_N^\perp)\left(\text{II}\left(\widetilde{\frac{\partial \Phi}{\partial s}}, \widetilde{\frac{\partial \Phi}{\partial s}}\right)\right) \equiv 0$ . Hence the image  $(\Phi)$  is totally geodesic. It follows from Part 1 that image  $(\Phi)$  has constant curvature 1.

To prove Part 5, we let  $\{J_1^*, \dots, J_{k-1}^*\}$  be any  $k-1$  linearly independent Jacobi fields in  $\mathcal{T}_N^*$ . It follows from Part 1 that for all  $i$ ,

$$\sec(\text{grad}\{\text{dist}(N_0, \cdot)\}, J_i^*) \equiv 1.$$

Together with Part 2 and our hypothesis that  $\text{Ric}_k(\dot{\gamma}, \cdot) \geq k$ , we conclude that for all  $J^* \in \mathcal{Z}_N^*$ ,  $\sec(\text{grad}\{\text{dist}(N_0, \cdot)\}, J^*) \geq 1$ .

It follows from Part 2 that  $\mathcal{Z}_N^* \oplus \mathcal{T}_N^*$  is a splitting of  $\Lambda_{N_0}$  into orthogonal, invariant subspaces for  $R(\cdot, \text{grad}\{\text{dist}(N_0, \cdot)\})\text{grad}\{\text{dist}(N_0, \cdot)\}$ . So

$$\sec(\text{grad}\{\text{dist}(N_0, \cdot)\}, Y) \geq 1$$

for all vectors  $Y$  orthogonal to  $\text{grad}\{\text{dist}(N_0, \cdot)\}$ .  $\square$

#### 4. THE STRUCTURE OF THE FOCAL SET

This section begins with a review of Hebda's notion of regular tangent focal points that generalizes a notion of Warner for conjugate points ([16], [29]). We next exploit the rigidity in Proposition 3.3 to show that every tangent focal point at time  $\frac{\pi}{2}$  is regular. This allows us to apply a result of Hebda and conclude that our focal set  $F \equiv \exp_N^\perp(S(N_0, \frac{\pi}{2}))$  is a smooth submanifold of  $M$ . The rigid structure also yields that  $F$  has focal radius  $\frac{\pi}{2}$ . We then further the analogy between the pair  $(N, F)$  and the dual sets in the proof of the Diameter Rigidity Theorem by showing that  $F$  (resp.  $N$ )

is the base of a Riemannian submersion from the unit normal sphere to any point of  $N$  (resp.  $F$ ).

**Definition 4.1.** ([16], cf [29]) *A tangent focal point  $v \in \nu(N)$  is called regular if and only if there is a neighborhood  $U$  of  $v$  so that every ray in  $\nu(N)$  that intersects  $U$  has at most one tangent focal point in  $U$ , not counting multiplicities. Otherwise  $v$  is called singular.*

Continuity of the curvature tensor implies that every  $v \in \nu(N)$  has a neighborhood  $U$  so that every ray meeting  $U$  has the same number of tangent focal points, counting multiplicities. So if  $v$  is a regular tangent focal point, then every ray  $tu$  in  $\nu(N)$  that intersects  $U$  has exactly one focal point  $t_0u$ , and the multiplicities of  $t_0u$  and  $v$  coincide. Thus regular tangent focal points have locally maximal order. Using this and ideas of [29], Hebda showed the following.

**Theorem 4.2.** ([16], cf [29]) *The set of regular tangent focal points is a smooth codimension 1 submanifold of  $\nu(N)$  that is an open, dense subset of the set of all tangent focal points.*

On  $\text{reg}_N \subset \nu(N)$ , we set

$$X \equiv \text{grad}(\text{dist}(N_0, \cdot)).$$

Along a fixed geodesic, focal points are isolated, so it follows that the set of regular, first-tangent focal points is an open, dense subset of the set of first-tangent focal points. It follows from the Gauss Lemma that  $\ker(D \exp_N^\perp)_X \perp X$ . Since the first-tangent focal set of our  $N \subset M$  is  $S(N_0, \frac{\pi}{2})$ , it follows that  $\ker(D \exp_N^\perp) \subset TS(N_0, \frac{\pi}{2})$ . Combining this with the Rank Theorem we get

**Corollary 4.3.** *Let  $\tilde{F}_{\text{reg}}$  be the set of regular first-tangent focal points, and let*

$$F_{\text{reg}} \equiv \exp_N^\perp(\tilde{F}_{\text{reg}}).$$

*Then  $F_{\text{reg}}$  is a smooth submanifold of  $M$  that is open and dense inside of  $F$ .*

**Lemma 4.4.** *Let  $v \in \nu(N)$  be a singular tangent focal point. For every neighborhood  $U$  of  $v$ , there is a regular tangent focal point  $w \in U$  so that*

$$\dim(\ker(D \exp_N^\perp)_w) < \dim(\ker(D \exp_N^\perp)_v).$$

*Proof.* Let  $U$  be any neighborhood of  $v$ . Replacing  $U$  with a possibly smaller neighborhood, we may assume that the total multiplicity of the focal points on each ray that intersects  $U$  is constant, and that the ray  $tv$  contains only one focal point in  $U$ . Since  $v$  is singular,  $U$  contains a ray with more than one focal point  $w_1 \neq w_2$ , which by hypothesis is not the ray through  $v$ . Since the multiplicity of the focal points in  $tw_1 \cap U$  and  $tv \cap U$  is the same, it follows that

$$\dim(\ker(D \exp_N^\perp)_{w_1}) < \dim(\ker(D \exp_N^\perp)_v). \quad (4.4.1)$$

It might be that  $w_1$  is not regular; however, since 4.4.1 holds for some  $w_1$  in any neighborhood of  $v$ , by repeating this argument a finite number of times, we get the desired conclusion.  $\square$

Let  $\gamma$  be a unit speed geodesic that leaves  $N$  orthogonally at time 0 with  $\gamma\left(\frac{\pi}{2}\right) \in F_{\text{reg}}$ . We set

$$\begin{aligned}\mathcal{Z} &\equiv \left\{ J \in \Lambda_N \mid J(0) = J\left(\frac{\pi}{2}\right) = 0 \right\}, \\ \mathcal{T}_N &\equiv \left\{ J \in \Lambda_N \mid J(0) \in T_{\gamma(0)}N \text{ and } J'(0) = S_{\gamma'(0)}(J(0)) \right\}, \text{ and} \\ \mathcal{T}_{F_{\text{reg}}} &\equiv \left\{ J \mid J\left(\frac{\pi}{2}\right) \in T_{\gamma\left(\frac{\pi}{2}\right)}F_{\text{reg}} \text{ and } J'\left(\frac{\pi}{2}\right) = -S\left(J\left(\frac{\pi}{2}\right)\right) \right\},\end{aligned}$$

where  $S_{\gamma'(0)}$  in the definition of  $\mathcal{T}_N$  is the shape operator of  $N$  and  $S$  in the definition of  $\mathcal{T}_{F_{\text{reg}}}$  is the Riccati operator of  $\Lambda_N$ . The next lemma shows that the  $S$  in the definition of  $\mathcal{T}_{F_{\text{reg}}}$  is also the shape operator of  $F_{\text{reg}}$  with respect to  $\gamma'\left(\frac{\pi}{2}\right)$ .

**Lemma 4.5.** *For  $\gamma$  as above:*

1.  $\gamma'\left(\frac{\pi}{2}\right) \in \nu_{\gamma\left(\frac{\pi}{2}\right)}F_{\text{reg}}$ .
2. The  $N$ -Jacobi fields along  $\gamma$  are the  $F_{\text{reg}}$ -Jacobi fields along

$$\gamma^{-1} : t \mapsto \gamma\left(\frac{\pi}{2} - t\right).$$

3. The subspaces  $\mathcal{T}_N$  and  $\mathcal{T}_{F_{\text{reg}}}$  are rigid, that is,

$$\begin{aligned}\mathcal{T}_N &= \{ \cos t E \mid E \text{ is parallel and tangent to } N \text{ at time } 0 \}, \text{ and} \\ \mathcal{T}_{F_{\text{reg}}} &= \{ \sin t E \mid E \text{ is parallel and tangent to } F_{\text{reg}} \text{ at time } \frac{\pi}{2} \}.\end{aligned}$$

4. Writing  $\Lambda_N$  for the  $N$ -Jacobi fields along  $\gamma$ , we have orthogonal splittings

$$\begin{aligned}\Lambda_N &= \mathcal{T}_N \oplus \mathcal{T}_{F_{\text{reg}}} \oplus \mathcal{Z} \text{ and} \\ \mathcal{Z}_N &= \mathcal{T}_{F_{\text{reg}}} \oplus \mathcal{Z},\end{aligned}$$

where  $\mathcal{Z}_N$  is as in Equation 3.2.1.

*Proof.* Part 1 is a consequence of the Gauss Lemma and the fact that  $F \equiv \exp_N^\perp(S(N_0, \frac{\pi}{2}))$ . The space  $\Lambda_N$  of  $N$ -Jacobi fields along  $\gamma$  are precisely the variation fields of variations by geodesics that leave  $N$  orthogonally at time 0. Similarly, the space  $\Lambda_{F_{\text{reg}}}$  of  $F_{\text{reg}}$ -Jacobi fields along  $\gamma$  are precisely the variation fields of variations by geodesics that arrive at  $F_{\text{reg}}$  orthogonally at time  $\frac{\pi}{2}$ . It follows from Part 1 that  $\Lambda_N \subset \Lambda_{F_{\text{reg}}}$ . Since  $\dim(\Lambda_N) = n - 1 = \dim(\Lambda_{F_{\text{reg}}})$ ,  $\Lambda_N = \Lambda_{F_{\text{reg}}}$ . This proves Part 2.

Since  $\gamma$  has no focal points for  $N$  on  $(0, \frac{\pi}{2})$ , it follows from Part 2 that  $\gamma^{-1}(t) = \gamma(\frac{\pi}{2} - t)$  has no focal points for  $F_{\text{reg}}$  on  $(0, \frac{\pi}{2})$ . By Part 5 of Proposition 3.3, all the radial sectional curvatures along  $\gamma$  are  $\geq 1$ . Thus Parts 3 and 4 follow from Parts 1 and 2 of Proposition 3.3 and the fact that  $\gamma^{-1}(t) = \gamma(\frac{\pi}{2} - t)$  has no focal points for  $F_{\text{reg}}$  on  $(0, \frac{\pi}{2})$ .  $\square$

**Lemma 4.6.**  $F_{\text{reg}} = F$ .

*Proof.* We set

$$F_{\text{sng}} \equiv F \setminus F_{\text{reg}},$$

and suppose, by way of contradiction, that  $F_{\text{sng}} \neq \emptyset$ .



Let  $\gamma_{\text{reg}}$  and  $\gamma_{\text{sng}}$  be geodesics that leave  $N$  orthogonally at time 0 with

$$\gamma_{\text{reg}}\left(\frac{\pi}{2}\right) \in F_{\text{reg}} \text{ and } \gamma_{\text{sng}}\left(\frac{\pi}{2}\right) \in F_{\text{sng}}.$$

The idea of the proof is to examine how the splitting  $\mathcal{Z}_N = \mathcal{T}_{F_{\text{reg}}} \oplus \mathcal{Z}$  behaves as a sequence of  $\gamma_{\text{reg}}$ 's approaches  $\gamma_{\text{sng}}$ . In particular, by Lemma 4.5,  $\mathcal{T}_{F_{\text{reg}}}$  is spanned by constant curvature 1 Jacobi fields. By continuity,  $\gamma_{\text{sng}}$  inherits such a family, and this forces  $\frac{\pi}{2}\gamma'_{\text{sng}}(0)$  to actually be regular. The details follow.

By appealing to Lemma 4.4, we can assume that

$$\dim\left(\ker\left(D\exp_N^\perp\right)_{\frac{\pi}{2}\gamma'_{\text{reg}}(0)}\right) < \dim\left(\ker\left(D\exp_N^\perp\right)_{\frac{\pi}{2}\gamma'_{\text{sng}}(0)}\right). \quad (4.6.1)$$

For either  $\gamma_{\text{reg}}$  or  $\gamma_{\text{sng}}$  we have the four spaces of Jacobi fields,  $\Lambda_N$ ,  $\mathcal{T}_N$ ,  $\mathcal{Z}_N$ , and  $\mathcal{Z}$ . We will distinguish the versions of the spaces along  $\gamma_{\text{reg}}$  from those along  $\gamma_{\text{sng}}$  with the superscripts  $\text{reg}$  and  $\text{sng}$ . When no superscript is present, the statement applies to either case.

For either  $\gamma_{\text{reg}}$  or  $\gamma_{\text{sng}}$ ,

$$\ker\left(D\exp_N^\perp\right)_{\frac{\pi}{2}\gamma'(0)} = \{J(0) \mid J \in \mathcal{T}_N\} \oplus \{J'(0) \mid J \in \mathcal{Z}\}. \quad (4.6.2)$$

Thus

$$\begin{aligned} \dim\left(\ker\left(D\exp_N^\perp\right)_{\frac{\pi}{2}\gamma'(0)}\right) &= \dim(\mathcal{T}_N) + \dim(\mathcal{Z}) \\ &= \dim(N) + \dim(\mathcal{Z}). \end{aligned}$$

Since  $\dim\left(\ker\left(D\exp_N^\perp\right)_{\frac{\pi}{2}\gamma'_{\text{sng}}(0)}\right) > \dim\left(\ker\left(D\exp_N^\perp\right)_{\frac{\pi}{2}\gamma'_{\text{reg}}(0)}\right)$ , the dimensions of  $\mathcal{Z}^{\text{reg}}$  and  $\mathcal{Z}^{\text{sng}}$  satisfy

$$\dim(\mathcal{Z}^{\text{sng}}) > \dim(\mathcal{Z}^{\text{reg}}). \quad (4.6.3)$$

Along  $\gamma_{\text{reg}}$ , Lemma 4.5 gives us an orthogonal splitting

$$\mathcal{Z}_N^{\text{reg}} = \mathcal{T}_{F_{\text{reg}}}^{\text{reg}} \oplus \mathcal{Z}^{\text{reg}}, \quad (4.6.4)$$

where

$$\mathcal{T}_{F_{\text{reg}}} = \left\{ \sin t E \mid E \text{ is parallel and tangent to } F_{\text{reg}} \text{ at time } \frac{\pi}{2} \right\}. \quad (4.6.5)$$

Combined with Inequality 4.6.3, this gives

$$\begin{aligned} \dim(\mathcal{Z}^{\text{sng}}) &> \dim(\mathcal{Z}^{\text{reg}}) \\ &= \dim(\mathcal{Z}_N^{\text{reg}}) - \dim\left(\mathcal{T}_{F_{\text{reg}}}^{\text{reg}}\right), \text{ by 4.6.4} \\ &= (n-1) - \dim(N) - \dim\left(\mathcal{T}_{F_{\text{reg}}}^{\text{reg}}\right). \end{aligned} \quad (4.6.6)$$

Note that  $\gamma_{\text{sng}}$  is a limit of  $\gamma_{\text{reg}}$ 's that satisfy 4.6.1. Further note that a  $J \in \mathcal{T}_{F_{\text{reg}}}^{\text{reg}}$  together with  $\gamma'_{\text{reg}}$  spans a plane of constant curvature 1. Thus by continuity,  $\mathcal{Z}_N^{\text{sng}}$  contains a subspace  $\mathcal{T}^{\text{sng}}$  of the form

$$\mathcal{T}^{\text{sng}} = \{\sin t E \mid E \text{ is parallel}\} \subset \mathcal{Z}_N^{\text{sng}} \setminus \mathcal{Z}^{\text{sng}}$$

with

$$\dim(\mathcal{T}^{\text{sng}}) = \dim(\mathcal{T}_{F_{\text{reg}}}^{\text{reg}}). \quad (4.6.7)$$

Moreover, by Remark 2.6 and Part 2 of Lemma F,  $\Lambda_N^{\text{sng}}$  splits orthogonally with one factor being  $\mathcal{T}^{\text{sng}}$ . Since  $\mathcal{T}^{\text{sng}}$  is a subspace of  $\mathcal{Z}_N^{\text{sng}}$ , we get

$$\mathcal{Z}_N^{\text{sng}} = \mathcal{T}^{\text{sng}} \oplus \mathcal{U}^{\text{sng}}, \quad (4.6.8)$$

where  $\mathcal{U}^{\text{sng}}$  is a space of Jacobi fields in  $\mathcal{Z}_N^{\text{sng}}$  that is orthogonal to  $\mathcal{T}^{\text{sng}}$  throughout  $(0, \frac{\pi}{2})$ .

The splitting 4.6.8 combined with  $\mathcal{T}^{\text{sng}} = \{\sin t E \mid E \text{ is parallel}\}$  gives that  $\mathcal{Z}^{\text{sng}}$  is a subspace of  $\mathcal{U}^{\text{sng}}$ , so

$$\begin{aligned} \dim(\mathcal{Z}^{\text{sng}}) &\leq \dim(\mathcal{U}^{\text{sng}}) \\ &= \dim(\mathcal{Z}_N^{\text{sng}}) - \dim(\mathcal{T}_{F_{\text{reg}}}^{\text{reg}}), \text{ by 4.6.8 and 4.6.7} \\ &= (n-1) - \dim(N) - \dim(\mathcal{T}_{F_{\text{reg}}}^{\text{reg}}). \end{aligned}$$

Since this contradicts Inequality 4.6.6, the result is proven.  $\square$

**Lemma 4.7.** *F is a totally geodesic closed submanifold of M with focal radius  $\frac{\pi}{2}$ .*

*Proof.* Since  $F_{\text{reg}} = F$ , it is a submanifold. Since  $F = \exp_N^\perp(S(N_0, \frac{\pi}{2}))$ , it is closed. It follows from Part 2 of Lemma 4.5 that  $\Lambda_N = \Lambda_F$ . Therefore the focal radius of  $F$  is  $\frac{\pi}{2}$ .

We have  $F = F_{\text{reg}}$ , so from Part 3 of Lemma 4.5,

$$\mathcal{T}_F = \left\{ \sin t E \mid E \text{ is parallel and tangent to } F \text{ at time } \frac{\pi}{2} \right\}.$$

In particular, for  $J \in \mathcal{T}_F$ ,  $J'(\frac{\pi}{2}) = 0$ . So  $F$  is totally geodesic.  $\square$

The next result will lead to the spherical rigidity portion of the conclusion of Theorem B, and also gives us Corollary G.

**Theorem 4.8.** *M has constant curvature 1 if either of the following holds:*

1. *F is not connected.*
2.  *$\dim(F) + \dim(N) = \dim(M) - 1$ .*

*Proof.* We have that  $\exp^\perp : S(N_0, \frac{\pi}{2}) \rightarrow F$  is onto. So if  $F$  is not connected, then  $S(N_0, \frac{\pi}{2})$  is not connected, and it follows that  $N$  is codimension 1 and has a trivial normal bundle. Since

$$\dim(F) + \dim(N) \leq \dim(\Lambda_N) = \dim(M) - 1,$$

to prove Part 1, it is enough to prove Part 2.

In general, we have an orthogonal splitting of  $\Lambda_N = \mathcal{T}_N \oplus \mathcal{T}_F \oplus \mathcal{Z}$  along any one of our normal geodesics. Since  $\dim(\mathcal{T}_F) = \dim(F)$ ,  $\dim(\mathcal{T}_N) = \dim(N)$ , and  $\dim(F) + \dim(N) = \dim(M) - 1$ ,  $\mathcal{Z} = 0$ , and our geodesic is spanned by constant curvature 1 Jacobi fields. Moreover,

$$\mathcal{T}_N = \mathcal{Z}_F \text{ and } \mathcal{T}_F = \mathcal{Z}_N.$$

Combining this with Part 1 of Proposition 3.3, it follows that along a geodesic leaving  $F$  orthogonally at time 0,

$$\mathcal{Z}_F = \{\sin tE \mid E \text{ is parallel}\}, \quad (4.8.1)$$

and along a geodesic leaving  $N$  orthogonally at time 0,

$$\mathcal{Z}_N = \{\sin tE \mid E \text{ is parallel}\}. \quad (4.8.2)$$

It follows from Equation 4.8.1 that for all  $x \in F$  and all  $r \in (0, \frac{\pi}{2})$ , the intrinsic metrics on

$$S_F(x, r) \equiv \exp_x^\perp \{v \in \nu_x(F) \mid |v| = r\} \quad (4.8.3)$$

are locally isometric to  $S^{\dim N}(\sin r)$ , that is, to the sphere of radius  $\sin r$  in  $\mathbb{R}^{\dim N+1}$ . Similarly, it follows that for all  $x \in N$  and all  $r \in (0, \frac{\pi}{2})$ , the intrinsic metrics on

$$S_N(x, r) \equiv \exp_x^\perp \{v \in \nu_x(N) \mid |v| = r\} \quad (4.8.4)$$

are locally isometric to  $S^{\dim F}(\sin r)$ . Since  $\mathcal{T}_N \oplus \mathcal{Z}_N$  is an orthogonal splitting, if  $\gamma$  leaves  $N$  orthogonally at time 0, then

$$S_N(\gamma(0), r) \text{ and } S_F\left(\gamma\left(\frac{\pi}{2}\right), \frac{\pi}{2} - r\right) \text{ intersect orthogonally at } \gamma(r). \quad (4.8.5)$$

Let

$$S_N(r) \equiv \exp_N^\perp \{v \in \nu(N) \mid |v| = r\},$$

and let  $\text{II}_r$  be the second fundamental form of  $S_N(r)$ , that is

$$\text{II}_r(U, V) \equiv g(\nabla_U V, \gamma'(r)),$$

where  $\gamma$  leaves  $N$  orthogonally at time 0.

Combining 4.8.1, 4.8.2 and 4.8.5, we have for  $Y \in \mathcal{T}_N$ , and  $W \in \mathcal{Z}_N$ ,

$$\begin{aligned} \text{II}_r(Y, Y) &= |Y|^2 \tan(r) \\ \text{II}_r(W, W) &= -|W|^2 \cot(r), \text{ and} \\ \text{II}_r(Y, W) &= 0. \end{aligned} \quad (4.8.6)$$

Now view  $\mathbb{S}^n$  as a join,  $\mathbb{S}^n = \mathbb{S}^{\dim N} * \mathbb{S}^{\dim F}$ , and let  $\tilde{\gamma}$  be a geodesic that leaves  $\mathbb{S}^{\dim N}$  orthogonally at time 0. Setting  $\tilde{M} \equiv \mathbb{S}^n$ ,  $\tilde{N} \equiv \mathbb{S}^{\dim N}$ , and  $\tilde{F} \equiv \mathbb{S}^{\dim F}$ , observe that 4.8.3, 4.8.4, 4.8.5, and 4.8.6 hold with  $M$ ,  $N$ , and  $F$  replaced by  $\tilde{M}$ ,  $\tilde{N}$ , and  $\tilde{F}$ . Observe further that Equations 4.8.3, 4.8.4, 4.8.5, and 4.8.6 together with the Gauss, Radial, and Codazzi-Mainardi Equations ([22]) determine the curvature tensor of  $\tilde{M} \equiv \mathbb{S}^n$ . Similarly, they determine the curvature tensor of  $M$ . Thus  $M$  has constant curvature 1.  $\square$

Throughout the remainder of Part 2, we assume that  $F$  is connected and

$$\dim(F) + \dim(N) \leq \dim(M) - 2. \quad (4.8.7)$$

**Lemma 4.9.** *1. Let  $x \in N$ . With respect to the constant curvature 1 metric on the unit normal sphere,  $\nu_x^1(N)$ , the map*

$$\begin{aligned} \pi_x &: \nu_x^1(N) \longrightarrow F \\ \pi_x &: v \mapsto \exp_N\left(\frac{\pi}{2}v\right) \end{aligned}$$

is a Riemannian submersion onto  $F$ .

2. Let  $x \in F$ . With respect to the constant curvature 1 metric on the unit normal sphere,  $\nu_x^1(F)$ , the map

$$\begin{aligned}\pi_x &: \nu_x^1(F) \longrightarrow N \\ \pi_x &: v \mapsto \exp_F\left(\frac{\pi}{2}v\right)\end{aligned}$$

is a Riemannian submersion onto  $N$ .

*Proof.* The proofs are identical, except for notation. We give the details for Part 1.

Let  $\gamma$  be a geodesic that leaves  $N$  orthogonally at time 0. Then

$$\begin{aligned}T_{\gamma'(0)}(\nu_{\gamma(0)}^1(N)) &= \{J'(0) \mid J \in \mathcal{Z}_N\}, \text{ and} \\ D\pi_{\gamma(0)}(J'(0)) &= J\left(\frac{\pi}{2}\right)\end{aligned}\tag{4.9.1}$$

for all  $J \in \mathcal{Z}_N$ .

Since  $\mathcal{T}_F \subset \mathcal{Z}_N$ , the splittings  $\Lambda_N = \mathcal{T}_N \oplus \mathcal{Z}_N = \mathcal{T}_N \oplus \mathcal{T}_F \oplus \mathcal{Z}$  give us an orthogonal splitting

$$\mathcal{Z}_N = \mathcal{Z} \oplus \mathcal{T}_F.$$

Combined with Equation 4.9.1 this gives an orthogonal splitting

$$T_{\gamma'(0)}(\nu_{\gamma(0)}^1(N)) = \{J'(0) \mid J \in \mathcal{Z}\} \oplus \{J'(0) \mid J \in \mathcal{T}_F\}\tag{4.9.2}$$

into the vertical and horizontal spaces, respectively, for  $\pi_{\gamma(0)}$ . Since  $\dim(\mathcal{T}_F) = \dim(F)$ , it follows from Equation 4.9.1 that  $\pi_{\gamma(0)}$  is a submersion. By Part 1 of Proposition 3.3, with  $F$  playing the role of  $N$ , the restriction of  $D\pi_{\gamma(0)}$  to the second summand in Equation 4.9.2 is an isometry. Thus  $\pi_{\gamma(0)}$  is a Riemannian submersion.  $\square$

## 5. THE SIMPLY CONNECTED CASE

Let  $\pi : \tilde{M} \longrightarrow M$  be the universal cover of  $M$ . Then each component  $\pi^{-1}(N)$  is a submanifold with focal radius  $\frac{\pi}{2}$  and dimension at least  $k$ . In particular,  $\tilde{M}$  contains a closed, connected, embedded submanifold with focal radius  $\frac{\pi}{2}$  and dimension at least  $k$ . So to prove Theorem B, it suffices to consider the case when  $M$  is simply connected and  $N$  is connected.

In this section, we will combine our simply connected hypothesis with Hebda's theorem on "very regular" focal loci. This will allow us to assert that, topologically,  $M$  is the union of two disk bundles, and the fibers of our Riemannian submersions,

$$\begin{aligned}\pi_x &: \nu_x^1(N) \longrightarrow F \\ \pi_x &: v \mapsto \exp_N\left(\frac{\pi}{2}v\right)\end{aligned}$$

and

$$\begin{aligned}\pi_x &: \nu_x^1(F) \longrightarrow N \\ \pi_x &: v \mapsto \exp_F\left(\frac{\pi}{2}v\right)\end{aligned}$$

are connected. We start with a review of Hebda's result.

**Definition 5.1.** (Hebda, [16]) Consider geodesics  $\gamma$  that leave  $N$  orthogonally at time 0.  $N$  has a very regular first focal locus if the multiplicity of the first focal point is independent of  $\gamma$  and, in case the multiplicity is one,  $\ker(D \exp_N^\perp)$  is contained in the tangent space to the tangent focal locus at every first-tangent focal point.

Along a geodesic that leaves our  $N$  orthogonally at time 0, the multiplicity of the focal point at time  $\frac{\pi}{2}$  is

$$\begin{aligned} \dim \mathcal{Z}_F &= \dim(\Lambda_N) - \dim \mathcal{T}_F \\ &= \dim(M) - 1 - \dim(F), \end{aligned}$$

and hence is constant. Since the focal radius of  $N$  along every geodesic is  $\frac{\pi}{2}$ , it follows from the Gauss Lemma that our  $N$  has a very regular first focal locus. Therefore, since  $M$  is simply connected, we can apply the following result of Hebda. (See Theorem 3.1 in [16] and the first line of its proof.)

**Theorem 5.2.** (Hebda, [16]) Suppose  $M$  is a connected, compact Riemannian manifold, and  $N$  is a connected, compact submanifold having a very regular first focal locus such that the inclusion  $\iota : N \hookrightarrow M$  induces a surjection of fundamental groups.

If the multiplicity of the first focal points of  $N$  is  $s - 1$ , then the first focal locus  $F$  of  $N$  in  $M$  is a submanifold of codimension  $s$  that coincides with the cut locus of  $N$  in  $M$ . Moreover, the tangent cut locus of  $N$  coincides with the first-tangent focal locus of  $N$ , and  $M$  is the union of two disk bundles

$$M = D_N \cup_\varphi D_F$$

over  $N$  and  $F$  respectively, where

$$\varphi : \partial D_N \longrightarrow \partial D_F$$

is a diffeomorphism.

By combining transversality and Theorem 5.2 we get following.

**Corollary 5.3.** Suppose that  $M$  is simply connected.

1. If  $\text{codim}(F) \geq 3$ , then  $N$  is simply connected.
2. If  $\text{codim}(N) \geq 3$ , then  $F$  is simply connected.

*Proof.* The two statements have dual proofs. We give the details for Part 1.

Transversality gives us

$$\pi_1(M) \cong \pi_1(M \setminus F),$$

and by Theorem 5.2,  $M \setminus F$  deformation retracts to  $N$ . □

Similarly, the cut locus statements in Theorem 5.2 gives us

**Corollary 5.4.** If  $M$  is simply connected, then  $\exp_N^\perp$  is injective on  $B(N_0, \frac{\pi}{2})$ .

**Lemma 5.5.** Let  $M$  be simply connected.

1. If  $\text{codim}(N) \geq 3$ , then the Riemannian submersions

$$\nu_x^1(N) \longrightarrow F, \quad x \in N$$

have connected fibers with positive dimension.

2. If  $\text{codim}(F) \geq 3$ , then the Riemannian submersions

$$\nu_x^1(F) \longrightarrow N, \quad x \in F$$

have connected fibers with positive dimension.

*Proof.* Since we have assumed that  $\dim(F) + \dim(N) \leq \dim(M) - 2$ ,

$$\begin{aligned} \dim(\nu_x^1(N)) &= \dim(M) - \dim(N) - 1 \\ &> \dim(F). \end{aligned}$$

Thus the fibers of  $\nu_x^1(N) \longrightarrow F$  have positive dimension.

By Corollary 5.3,  $F$  is simply connected if  $\text{codim}(N) \geq 3$ . In this case, the long exact homotopy sequence for  $\nu_x^1(N) \longrightarrow F, x \in N$  gives

$$\pi_1(F) \longrightarrow \pi_0(\text{fiber}) \longrightarrow 0,$$

since  $\pi_0(\nu_x^1(N))$  is trivial. Thus the first conclusion holds. A similar argument gives us the second conclusion, if  $\text{codim}(F) \geq 3$ .  $\square$

Since we have assumed that

$$\dim(F) + \dim(N) \leq \dim(M) - 2, \tag{5.5.1}$$

$\text{codim}(N) \geq 2$ . Since  $\dim(N) \geq 1$ , we have  $\text{codim}(F) \geq 3$ . Combining this with Lemma 4.7, Corollary 5.4, and Lemma 5.5, we have

**Theorem 5.6.** *Let  $M$  be a complete Riemannian  $n$ -manifold with  $\text{Ric}_k \geq k$  and  $N$  any closed, connected, submanifold of  $M$  with  $\dim(N) \geq k$  and focal radius  $\frac{\pi}{2}$ . If  $M$  is simply connected and not isometric to the unit sphere, then:*

1.

$$\dim(N) + \dim(F) \leq n - 2.$$

2.  $N$  is totally geodesic and isometric to an even dimensional CROSS.

3. The focal set  $F$  of  $N$  is totally geodesic and is either a point or is isometric to an even dimensional CROSS.

4. The normal exponential maps of  $N$  and  $F$  are injective on the  $\frac{\pi}{2}$ -balls around the zero sections of the normal bundles of  $N$  and  $F$ .

5. The conclusions of Proposition 3.3 hold with  $N$  replaced by  $F$ .

6. For every  $x \in F$  the map

$$\begin{aligned} \pi_x &: \nu_x^1(F) \longrightarrow N \\ \pi_x &: v \mapsto \exp_N\left(\frac{\pi}{2}v\right) \end{aligned}$$

is a Riemannian submersion whose fibers are connected and have positive dimension.

7. For every  $x \in N$  the map

$$\begin{aligned} \pi_x &: \nu_x^1(N) \longrightarrow F \\ \pi_x &: v \mapsto \exp_N\left(\frac{\pi}{2}v\right) \end{aligned}$$

is a Riemannian submersion whose fibers are connected and have positive dimension.

## 6. RIGIDITY IN THE SECTIONAL CURVATURE CASE

In this section, we complete the proof of Theorem B in the case when the sectional curvature of  $M$  is  $\geq 1$ . Since  $M$  is simply connected, by Part 4 of Theorem 5.6, the diameter of  $M$  is  $\geq \frac{\pi}{2}$ . So combining the Diameter Rigidity Theorem with our dimension hypothesis 5.5.1 and Theorem 5.6 gives us the following.

**Proposition 6.1.** *If  $M$  does not have constant curvature 1, then the following hold.*

1.  *$M$  is isometric to a compact, rank one, symmetric space or is homeomorphic to  $S^n$ .*
2.  *$N$  is even dimensional and isometric to the base of a Hopf fibration.*
3.  *$F$  is either a point or is isometric to the base of a Hopf fibration and is even dimensional*

To conclude the proof of Theorem B, we show that conclusions 2 and 3 are not compatible with  $M$  being a topological sphere.

If  $M$  is a sphere, the long exact homology sequence of the pair  $(M, F)$  gives

$$H_q(M, F) \cong H_{q-1}^\#(F)$$

for  $q \leq n-1$ .

Use Theorem 5.2 to write

$$M = D_N \cup_\varphi D_F.$$

By excision,

$$H_q(M, F) \cong H_q(D_N, \partial D_N).$$

Thus for  $q \leq n-1$ ,

$$H_{q-1}^\#(F) \cong H_q(D_N, \partial D_N). \tag{6.1.1}$$

By Proposition 6.1,  $F$  is either an even dimensional CROSS or a point, and  $N$  is an even dimensional CROSS. It follows from Equation 6.1.1 that  $H_q(D_N, \partial D_N) \cong 0$  if  $q$  is even and  $\leq n-1$ . If  $q$  is odd and  $q+1 \leq n-1$ , then the sequence of the pair  $(D_N, \partial D_N)$  gives

$$0 = H_{q+1}(D_N, \partial D_N) \longrightarrow H_q(\partial D_N) \longrightarrow H_q(N) = 0,$$

since  $N$  is an even dimensional CROSS. Thus

$$H_q(\partial D_N) \cong 0 \text{ if } q \text{ is odd and } \leq n-2. \tag{6.1.2}$$

Since  $N$  is isometric to the base of a Hopf fibration with connected fibers,  $\dim(N) \geq 2$ . Since  $\dim(N) + \dim(F) \leq n-2$ , we get  $n \geq 4$ . So  $\dim(\partial D_N) \geq 3$ . Since  $\partial D_N$  is a connected, compact, odd dimensional manifold, 6.1.2 implies, via Poincaré duality, that  $\partial D_N$  is a  $\mathbb{Z}_2$ -homology sphere of dimension  $\geq 3$ . Thus the Mayer-Vietoris sequence with  $q \in \{2, \dots, n-2\}$  yields

$$0 = H_q(\partial D_N) \longrightarrow H_q(D_N) \oplus H_q(D_F) \longrightarrow H_q(M) \longrightarrow H_q(\partial D_N) = 0.$$

Since  $D_N$  has the homotopy type of the CROSS  $N$ , and  $\dim(M) \geq \dim(N) + 2 \geq 4$ ,  $M$  cannot be homeomorphic to a sphere.

## 7. RIGIDITY AND INTERMEDIATE RICCI

In this section, we complete the proof of Theorem B. This is achieved by analyzing the radial geometry from  $N$  and  $F$ . Proposition 3.3, Lemma 4.5, and Lemma 4.7 give us rigid radial geometry along the distribution spanned by the Jacobi fields in  $\mathcal{T}_N$  and  $\mathcal{T}_F$ . To prove rigidity for the  $\mathcal{Z}$ -Jacobi fields, we show, in Proposition 7.4, that as in the proof of the Diameter Rigidity Theorem, there are enough other dual pairs in  $M$  to force the  $\mathcal{Z}$ -Jacobi fields to span projective lines. This is achieved via the next three results, wherein the hypotheses that  $N$  is connected and  $M$  is simply connected are still in force.

**Proposition 7.1.** *1. For any  $p \in N$  the cut point along any geodesic emanating from  $p$  is at distance  $\frac{\pi}{2}$  from  $p$ .*

*2. For any  $p, q \in N$ , any minimal geodesic of  $M$  between  $p$  and  $q$  lies entirely in  $N$ .*

*Proof.* Let  $v \in T_p M \setminus \{T_p N, \nu_p(N)\}$  be any unit vector. Let  $v^T$  and  $v^\perp$  be the unit vectors that point in the same directions as the projections of  $v$  onto  $T_p N$  and  $\nu_p(N)$ , respectively. By Part 4 of Proposition 3.3,  $\text{span}\{v^T, v^\perp\}$  exponentiates to a totally geodesic immersed surface  $\Sigma$  of constant curvature 1 that contains  $\gamma_v$ . By Corollary 5.4, the restriction of  $\exp_p$  to the interior of the circular sector

$$\text{Sect}\left(v^T, v^\perp, \frac{\pi}{2}\right) \equiv \left\{ \exp_p(t(v^T \cos s + v^\perp \sin s)) \mid t, s \in \left[0, \frac{\pi}{2}\right] \right\}$$

of radius and angle  $\frac{\pi}{2}$  spanned by  $v^T$  and  $v^\perp$  is an embedding. If  $w \in T_p M$  is not in  $T_p N, \nu_p(N)$ , or  $\text{span}\{v^T, v^\perp\}$ , then Corollary 5.4 gives that interiors of  $\text{Sect}(v^T, v^\perp, \frac{\pi}{2})$  and  $\text{Sect}(w^T, w^\perp, \frac{\pi}{2})$  are disjoint. It follows that the cut-time of any unit  $v \in T_p M \setminus \{T_p N, \nu_p(N)\}$  is  $\geq \frac{\pi}{2}$ , and by continuity, the cut-time of any unit  $v \in T_p M$  is  $\geq \frac{\pi}{2}$ .

Since the focal radius of  $N$  is  $\frac{\pi}{2}$ , to complete the proof of Part 1, it suffices to check that any unit  $v \in T_p M \setminus \nu_p(N)$  has a cut point at time  $\frac{\pi}{2}$ . Since  $N$  is a CROSS with curvature in  $[1, 4]$ , there is a unit vector  $w^T \in T_p(N)$  with

$$\gamma_{w^T}\left(\frac{\pi}{2}\right) = \gamma_{v^T}\left(\frac{\pi}{2}\right) \text{ and } w^T \neq v^T.$$

Let  $u \in \nu_{\gamma_{v^T}(\frac{\pi}{2})}(N)$  be a unit vector. Let  $U_v$  and  $U_w$  be the backwards parallel transports of  $u$  along  $\gamma_{v^T}$  and  $\gamma_{w^T}$ . Let  $\Sigma_{v^T}$  and  $\Sigma_{w^T}$  be the spherical sectors of radius and angle  $\frac{\pi}{2}$  obtained via Part 4 of Proposition 3.3 by exponentiating  $U_v$  and  $U_w$ . That is

$$\Sigma_{v^T} \equiv \left\{ \exp_N(tU_v(s)) \mid s, t \in \left[0, \frac{\pi}{2}\right] \right\}$$

and

$$\Sigma_{w^T} \equiv \left\{ \exp_N(tU_w(s)) \mid s, t \in \left[0, \frac{\pi}{2}\right] \right\}.$$

Then  $\Sigma_{w^T}$  and  $\Sigma_{v^T}$  are different surfaces, but since

$$U_v\left(\frac{\pi}{2}\right) = U_w\left(\frac{\pi}{2}\right) = u \in \nu_{\gamma_{v^T}(\frac{\pi}{2})}(N),$$

$\Sigma_{w^T}$  and  $\Sigma_{v^T}$  intersect along  $\exp_N(tu)$ . So every cut point from  $p$  occurs at distance  $\frac{\pi}{2}$  from  $p$ .



By Corollary 5.4, the intersection of  $N$  with any of the sectors  $\text{Sect}(v^T, v^\perp, \frac{\pi}{2})$  is precisely  $\gamma_{v^T}[0, \frac{\pi}{2}] \subset N$ . Part 2 follows.  $\square$

**Proposition 7.2.** *For any  $p \in N$ , the set of points in  $M$  at distance  $\frac{\pi}{2}$  from  $p$ ,*

$$A(p) \equiv S\left(p, \frac{\pi}{2}\right),$$

*is a closed submanifold of dimension*

$$\dim(N) + \dim F$$

*and focal radius  $\frac{\pi}{2}$ .*

*Proof.* Let

$$A_N(p) \equiv \left\{ x \in N \mid \text{dist}(p, x) = \frac{\pi}{2} \right\}.$$

Using the rigid hinges of Part 4 of Proposition 3.3 and the fact that  $A(p) \equiv S(p, \frac{\pi}{2})$  is the cut locus of  $p$  we see that we can describe  $A(p)$  in two ways:

$$A(p) = \left\{ \exp_N^\perp(tv) \mid v \in \nu^1(N)|_{A_N(p)} \text{ and } t \in \left[0, \frac{\pi}{2}\right] \right\},$$

and

$$A(p) = \left\{ \exp_F^\perp(tv) \mid v \in \nu^1(F), \exp_F\left(\frac{\pi}{2}v\right) \in A_N(p), \text{ and } t \in \left[0, \frac{\pi}{2}\right] \right\}. \quad (7.2.1)$$

Either description shows  $A(p)$  is compact. Since both  $N$  and  $F$  have focal radius  $\frac{\pi}{2}$ , both descriptions show, via Corollary 5.4, that  $A(p) \setminus \{F \cup N\}$  is a manifold. The first description shows that  $A(p)$  is smooth near  $N$ .

Recall that for every  $x \in F$ , the map

$$\begin{aligned} \pi_x &: \nu_x^1(F) \longrightarrow N, \\ \pi_x &: v \mapsto \exp_N\left(\frac{\pi}{2}v\right) \end{aligned}$$

is a Riemannian submersion with connected fibers. Combining this with Part 4 of Proposition 3.3 and Proposition 7.1, we rewrite 7.2.1 as

$$A(p) = \cup_{x \in F} \left\{ \exp_F^\perp(tv) \mid v \in \nu_x^1(F), v \perp \pi_x^{-1}(p), \text{ and } t \in \left[0, \frac{\pi}{2}\right] \right\}, \quad (7.2.2)$$

where  $\{v\}$  and  $\pi_x^{-1}(p)$  are subsets of  $\nu_x(F)$ , and the notion of perpendicular comes from the inner product that  $g$  induces on  $\nu_x(F)$ . This shows  $A(p)$  is smooth near  $F$ .

Next, we decompose  $\nu_x^1(F)$  as a join

$$\begin{aligned} \nu_x^1(F) &= \pi_x^{-1}(p) * (\pi_x^{-1}(p))^\perp, \text{ where} \\ (\pi_x^{-1}(p))^\perp &\equiv \left\{ v \in \nu_x^1(F) \mid v \perp \pi_x^{-1}(p) \right\}. \end{aligned}$$

Note that 7.2.2 gives that for any  $x \in F$ ,

$$\dim(A(p)) = \dim F + \dim(\pi_x^{-1}(p))^\perp + 1. \quad (7.2.3)$$

Since  $\pi_x: \nu_x^1(F) \longrightarrow N$  is a Riemannian submersion,

$$\dim(N) = \dim(H_v(\pi_x^{-1}(p))), \quad (7.2.4)$$

where  $H_v(\pi_x^{-1}(p))$  is the horizontal space for  $\pi_x$  at any  $v \in \pi_x^{-1}(p) \subset \nu_x^1(F)$ .

The join decomposition  $\nu_x^1(F) = \pi_x^{-1}(p) * (\pi_x^{-1}(p))^\perp$  identifies  $(\pi_x^{-1}(p))^\perp$  with the unit vectors in  $H_v(\pi_x^{-1}(p))$ . So

$$\dim(\pi_x^{-1}(p))^\perp = \dim(H_v(\pi_x^{-1}(p))) - 1.$$

Combining with Equations 7.2.3 and 7.2.4, we get

$$\dim(A(p)) = \dim F + \dim(N) - 1 + 1.$$

Since  $A(p) = S(p, \frac{\pi}{2})$  is a smooth submanifold, and every geodesic that leaves  $p$  has cut point at distance  $\frac{\pi}{2}$  from  $p$ , it follows from 1<sup>st</sup>-variation that every geodesic leaving  $p$  arrives orthogonally at  $A(p)$  at time  $\frac{\pi}{2}$ . This identifies the unit tangent sphere at  $p$ ,  $S_p$ , with the unit normal bundle of  $A(p)$ ,  $\nu^1(A(p))$ . Combined with Proposition 7.1, it follows that the focal radius of  $A(p)$  along any normal geodesic is greater than or equal to  $\frac{\pi}{2}$ . So it follows from Theorem A that the focal radius of  $A(p)$  is exactly  $\frac{\pi}{2}$ .  $\square$

It follows that Theorem 5.6 applies with  $N^p = A(p)$  and  $F^p = p$ .

Next, we apply Propositions 7.1 and 7.2 to  $q \in A(p)$  and, with a further application of Theorem 5.6, get the following result.

**Proposition 7.3.** *1. Every cut point from  $q$  occurs at distance  $\frac{\pi}{2}$  from  $q$ .  
2. The set of points in  $M$  at distance  $\frac{\pi}{2}$  from  $q$ ,*

$$A(q) \equiv S\left(q, \frac{\pi}{2}\right),$$

*is a closed submanifold with focal radius  $\frac{\pi}{2}$  and dimension  $\dim N + \dim F$ .*

*3.  $A(q)$  is totally geodesic and isometric to a CROSS.  
4. For any  $a, b \in A(q)$ , any minimal geodesic of  $M$  between  $a$  and  $b$  lies entirely in  $A(q)$ .*

Returning  $N^p = A(p)$  and  $F^p = p$ , we now have the following refinement of Proposition 3.3.

**Proposition 7.4.** *Let  $\gamma$  be a unit speed geodesic that leaves  $N^p$  orthogonally at time 0 and let  $\Lambda_{N^p}$ ,  $\mathcal{T}_{N^p}$  and  $\mathcal{Z}_{N^p}$  be as in 3.2.1.*

*1.  $\mathcal{Z}_{N^p} \oplus \mathcal{T}_{N^p}$  is a parallel, orthogonal splitting of  $\Lambda_{N^p}$  along  $(0, \frac{\pi}{2})$ .  
2.  $\mathcal{T}_{N^p}$  and  $\mathcal{Z}_{N^p}$  have the forms*

$$\begin{aligned} \mathcal{T}_{N^p} &\equiv \{\cos tE \mid E \text{ is parallel and tangent to } N^p \text{ at time } 0\}, \\ \mathcal{Z}_{N^p} &\equiv \left\{ \frac{1}{2} \sin 2tE \mid E \text{ is parallel and orthogonal to } N^p \text{ at time } 0 \right\}. \end{aligned}$$

*Proof.* Apart from the second equation in Part 2, this is a repeat of Parts 1 and 2 of Proposition 3.3. The second equation in Part 2 follows from Part 2 of Proposition 7.3. Indeed, let  $\gamma$  be a unit speed geodesic from  $x \in N^p$  to  $p$ . Choose  $q \in N^p$  at distance  $\frac{\pi}{2}$  from  $x$ , and apply Proposition 7.3 to  $q$ . It follows that  $\gamma \subset A(q)$ , and  $A(q)$  is a totally geodesic CROSS. In particular, the Jacobi fields along  $\gamma$  have the indicated form if they are tangent to  $A(q)$ . Since the normal space to  $A(q)$  along  $\gamma$  is spanned by a subspace of  $\mathcal{T}_{N^p}$ , the result follows.  $\square$

We finish the proof of Theorem B along the lines of the proof of Theorem 4.3 in [13] by using Cartan's Theorem (Theorem 2.1, page 157 of [6]).

Since Theorem 5.6 applies to  $N^p = A(p)$ , there is a CROSS  $P$  with

$$\dim(P) = \dim(M)$$

and

$$\dim(\mathcal{Z}_{N^p}) = \dim(\mathbb{F}) - 1,$$

where  $\mathbb{F}$  is the division algebra that defines  $P$ .

Choose a point  $\tilde{p} \in P$ . Since  $P$  is a CROSS, we have a Riemannian submersion

$$\tilde{\pi}_{\tilde{p}} : S_{\tilde{p}} \longrightarrow A(\tilde{p}) \equiv \left\{ x \in P \mid \text{dist}(x, P) = \frac{\pi}{2} \right\}$$

that is isometrically equivalent to a Hopf Fibration. Since  $\dim(\mathcal{Z}_{N^p}) = \dim(\mathbb{F}) - 1$ , we have, using [12] and [31], that  $\tilde{\pi}_{\tilde{p}}$  is isometrically equivalent to

$$\pi_p : S_p \longrightarrow N^p \equiv A(p).$$

Let

$$I : S_{\tilde{p}} \longrightarrow S_p$$

be a linear isometric equivalence between  $\tilde{\pi}_{\tilde{p}}$  and  $\pi_p$ . Then we have a commutative diagram

$$\begin{array}{ccc} S_{\tilde{p}} & \xrightarrow{I} & S_p \\ \downarrow \tilde{\pi}_{\tilde{p}} & & \downarrow \pi_p \\ A(\tilde{p}) & \xrightarrow{\hat{I}} & A(p) \end{array}$$

Since  $\tilde{\pi}_{\tilde{p}}$  and  $\pi_p$  are Riemannian submersions and  $I$  is an isometry,  $\hat{I}$  is an isometry.

Via the Cartan Theorem and Proposition 7.4, we see that  $\iota \equiv \exp_p \circ I \circ \exp_{\tilde{p}}^{-1}$  defines an isometry between  $P \setminus A(\tilde{p})$  and  $M \setminus A(p)$  that induces the isometry  $\hat{I} : A(\tilde{p}) \longrightarrow A(p)$ , and thus  $\iota$  extends to an isometry  $P \longrightarrow M$ . This completes the proof of Theorem B.

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